JOURNAL

OF THE

DEPARTMENT OF SCIENCE

University of Calcutta

Journal

of the

Department of Science

Vol. IX

CALCUTTA UNIVERSITY PRESS

1928

PRINTED AND PUBLISHED BY BHUPENDRA LALL BANERJEE AT THE
UNIVERSITY PRESS, SENATE HOUSE, CALCUTTA

C. U. Press, Reg. No. 308BJ—August—F

CONTENTS

ADHARCHANDRA MOOKERJEE LECTURES

BY

J. C. GHOSH, D.Sc.,

PROFESSOR, DACCA UNIVERSITY

1928

FIRST LECTURE [1]

Interaction between Matter and Radiation.

I shall speak to you this afternoon on recent investigations on the mechanism of light absorption by molecules and the chemical changes that result from this process. It is a subject which lies on the borderland between molecular physics and photo-chemistry, and the preliminary results that have been obtained up till now have been due to the closest co-operation between physicists and physical chemists. In the first place I should mention that the old idea, that all the molecules of an illuminated system consisting of one component absorb radiant energy equally, have been discarded altogether. It is now universally accepted that during the process of light absorption radiant energy is converted into the potential energy of an optical or a binding electron and this transformation takes place in a quantum switch. When a system is exposed to monochromatic light radiation ν, each molecule can get into an excited condition by absorbing an amount of energy $h\nu$ and if E be the energy of incident radiation per second, the number of molecules primarily affected per second is $E/h\nu$. Molecules in excess over the number $E/h\nu$ are not disturbed in the first instance. Recent investigations on photo-chemistry have chiefly been carried out with one object in view :—to trace the various phases through which such an excited molecule may pass before the whole of the absorbed radiant energy is degenerated into heat. We shall consider several possibilities.

It is possible that an electron may part company with the parent atom or molecule, and we meet with the process of ionisation. Such ionisation by absorption of light forms the province of a special branch of Physics—Photo-electricity ; and we need not pursue the discussion of the subject here.

In the case of a diatomic molecule, the absorption of radiant energy may be followed by the dissociation of the molecule into atoms. It is possible that the atoms so formed are not existing in the normal states where the optical electron has the minimum potential energy, but they may be in excited states where the electron is in the higher quantum orbits. Franck has shown how an examination

[1] Adharchandra Mookerjee Lecture delivered on 5th February, 1928.

of band absorption spectra can give evidence in favour of decomposition of molecules into atoms. The band absorption spectra of the photo-active gases iodine, bromine and chlorine have been very thoroughly investigated recently. Fig. I shows the most important

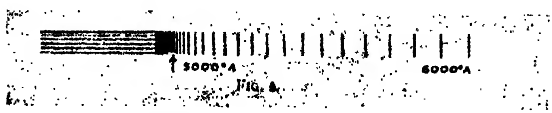


Fig. I.

Band Spectra of Iodine.

series of bands in the case of iodine. It will be seen that the shorter the wave-length the smaller is the interval between the oscillation quantum states and there appears to be a real convergence limit at 5,000Å, such as is found at the end of a line series, *e.g.*, a Balmer series for hydrogen, and similarly to that case also, a strong continuous spectra beyond that limit. According to Franck, this continuous absorption beyond 5,000Å indicates the dissociation of a molecule of iodine into two atoms. But both these atoms are not in normal states. One of them is in the normal state, while the other is in a metastable state.

The principal argument in favour of such a dissociation of the iodine molecule has been obtained from the study of the nature of fluorescence of iodine vapour. Fluorescence, as you all know, is due to re-emission of light when molecules which have been brought to higher quantum states by absorption of radiation revert to the normal state. Now Dymond has shown that emission of resonance lines by iodine vapour at very low pressures, on illumination with a series of wave-lengths from long waves to the neighbourhood of λ 5,000Å consists of molecular fluorescence. But wave-lengths shorter than λ 5,000Å give no trace of fluorescence although they are more strongly absorbed.

The second argument in favour of such dissociation into a normal and a metastable atom in the $2p_1$ state, as it is technically called, consists in the very close agreement between the heats of dissociation calculated from optical data with those obtained directly by physico-chemical experiments. Now the difference in energy between the

$2p_2$ state of the iodine atom and the metastable $2p_1$ state can be easily obtained if we can find out a recurring doublet separation in the spectra of the iodine atom. It is well-known that the frequency difference in a doublet multiplied by the Planck's constant h is a measure of the energy difference between the two p states ; even chemists are familiar with the existence of such doublet in the D line of sodium which really consists of two lines very close to each other. Turner has recently obtained a recurring doublet separation in the case of iodine atom of 7,600 cm.$^{-1}$ which he interprets as an accurate measure of this interval. This corresponds to an energy difference of 21,700 cal. between the normal and metastable states for a gram atom of iodine. When an iodine molecule absorbs light of wave-length λ 5,000Å, which is the convergence limit of band absorption, it gains an energy quantum of this mono-chromatic radiation, and it can be easily shown that for a gram molecule the total gain in energy is 57,000 calories ; absorption of 57,000 calories therefore gives us a gram atom of normal iodine and a gram atom of metastable iodine with excess energy of 21,700 cal. Hence the heat of dissociation of a gram molecule of iodine calculated from purely spectral data is 35,300 cal. whereas the observed experimental value of the energy is 34,500 cal. The doublet separation in the case of chlorine and bromine has not yet been directly observed, but the magnitude of such separation can be calculated from Lande's Rule, according to which in a family of related elements, the frequency difference in a doublet is proportional to the square of the nuclear charge. Hence for chlorine

$$\Delta\nu=(7600\text{cm.}^{-1})\times\frac{17^2}{53^2}$$

and for bromine

$$\Delta\nu=(7600\text{cm.})^{-1}\times\frac{35^2}{53^2}$$

Table I shows the energy difference between the normal and metastable p states, and the calc. and obs. heat of dissociation of halogens. The agreement is fairly good.

TABLE I.

Wave-length of the Band limit.	Energy of the Band limit.	Excess Energy of metastable state.	Calc. Energy of Dissociation.	Obs. Energy of Dissociation.
Iodine 5,000Å	57,000 cal.	21,700 cal.	35,300	34,500
Bromine 5,200Å	55,000 cal.	10,500 cal.	44,500	46,000
Chlorine 4,800Å	59,000 cal.	2,500 cal.	56,500	54,000

The third evidence in favour of the dissociation of halogen molecules by absorption of light shorter than green rays, was obtained by Bodenstein and Berthoud from the study of velocity of photo-chemical reactions long before this suggestion of Franck. I shall refer to a simple reaction which was first studied by Dhar.

$$\begin{array}{l} \mathrm{COOK} \\ | \\ \mathrm{COOK} \end{array} + I_2 \longrightarrow 2KI + 2CO_2$$

Dhar was of opinion that the rate of disappearance of iodine in presence of an excess of oxalate was proportional to the intensity of incident light. But Berthoud and Bellonet have very carefully studied this reaction and conclude that the velocity is proportional to the square root of intensity of incident light. This puzzling relation can be satisfactorily explained only on the assumption that a molecule of iodine breaks up on absorption of radiation into two atoms and that atomic iodine only can react with potassium oxalate. The concentration of atomic iodine at any time can be obtained from the consideration of a reversible reaction of the following type

$$I_2 + h\nu \longrightarrow 2I \quad \ldots \quad (1); \qquad I + I \longrightarrow I_2 \quad \ldots \quad (2)$$

Now if the absorption of incident radiation is practically complete, then the number of iodine molecules which absorb radiation per sec. is equal to $E/h\nu$ where E is the energy of incident radiation per sec. The rate of formation of iodine atoms is therefore proportional to $E/h\nu$ and the rate of recombination of iodine atoms, according to mass action law as applied to reaction (2) is $K[I]^2$. Hence under conditions of equilibrium,

$$E/h\nu = K[I]^2$$

or the concentration of iodine atom at any time $= K'\sqrt{E}$. Since the velocity of oxalate oxidation is proportional to the concentration of

iodine atom existing at any time, it should be proportional to the square root of incident intensity.

Similar experiments on the photobromination of stilbene and of cinnamic acid have shown that the velocities of such reactions are proportional to the square root of incident light energy and hence the inference is drawn that the molecules of bromine break up into atoms by absorption of light and that these atoms are responsible for keeping the action going.

The classical researches of Warburg have shown that the gases HI, HBr, HCl are decomposed by absorption of ultraviolet light into their constituent elements, and it was found that for each quantum absorbed two molecules undergo decomposition. The following mechanisms can explain equally well this experimental observation :

A. 1. $HI + h\nu \longrightarrow HI$ Excited)

2. $HI\ (Ex) + HI$ (Normal) $\longrightarrow H_2 + I_2$

B. or 1. $HI + h\nu \longrightarrow H + I$

2. $H + HI \longrightarrow H_2 + I$

3. $I + I \longrightarrow I_2$.

The reaction A. 2, raises a fundamental point—whether the quantum energy of excitation can, during a process of collision, be transformed into the chemical energy of the dissociation products. Possibility of such type of transformation has been definitely demonstrated, and it will be discussed when we consider the phenomena of photo-sensitisation. It has not yet been decided which of these two processes represent the real facts; Tingey and Gerke believe that the structure of the band absorption spectra of the gaseous halogen acids is actually continuous and suggest that the acid molecule decomposes into atoms. But our knowledge of the ultraviolet absorption band spectra of the gases is imperfect at this stage, and nothing definite can be said now.

I have spoken of the molecules decomposing into atoms by absorption of light. There is the other possibility of molecules passing into excited states by absorption of radiation. The most familiar example of such a reaction is to be found in the photographic dry plate where,

according to the researches of Eggert and Noddack, the following primary changes take place :

$$AgBr + h\nu \longrightarrow AgBr\ (E)$$

$$AgBr(E) + \text{gelatine} \longrightarrow Ag + \text{Brominated gelatine.}$$

For each quantum absorbed, one atom of silver is produced. On development, however, complicated catalytic processes appear which need not be discussed here. In liquid systems we may refer to a very simple case which has been studied by Rudeberg—the hydrolysis of mono-chloro-acetic acid. The absorption of one quantum of ultra-violet light of wave-length 3000-2000A excites a molecule of the acid which reacts with water to give HCl and CH_2OH—COOH.

We know that as a preliminary to excitation, an electron is removed from its normal position to a higher quantum state and that it sooner or later returns to the normal state with loss of energy by emission or collision with another body. The time which the electron spends on an average at the higher stationary state or, in other words, the life of the excited atom, has been the subject of extensive researches. Recent investigations have brought to light the fact that there are several types of excited states possible for a molecule, some of which have very short life period say 10^{-8} or 10^{-9} seconds while others may have life period as large as 10^{-2} seconds. Thus Wien from a study of canalrays finds that the stationary state of excited hydrogen atom, from which the quantum jump to a lower state is responsible for the Hα red line, has a life period of 10^{-8} seconds and that the stationary state responsible for the emission of the principal Hg line in the ultraviolet λ 2536 has also a life period of 9×10^{-8} seconds.

The value of life periods of excited states can also be determined by physico-chemical methods. The method will be clear if we enter into the mechanism of a photo-sensitised reaction. At ordinary temperature hydrogen gas does not react with copper-oxide to produce water. If hydrogen gas is illuminated with ultraviolet light, say 2536A, it does not reduce copper-oxide. Nor is this to be expected as hydrogen has got no absorption band in this region. If however the system contains mercury vapour and is illuminated by the Hg line

2536, hydrogen reacts with copper-oxide to produce water. Here the mechanism is obviously the following:

$$Hg + h\nu \longrightarrow Hg \text{ (excited)} \qquad \dots \quad (1)$$

$$Hg\,(E) + H_2 \longrightarrow Hg \text{ (normal)} + 2H \qquad \dots \quad (2)$$

$$2H + CuO \longrightarrow H_2O + Cu.$$

It will be noticed that in reaction (2) the quantum energy of the excited Hg atom where the electron has been pushed out from the normal S to a p level, is transformed into the energy of the dissociation of the hydrogen molecule. This type of inelastic collision is very frequent in all photo-chemical reactions. It has been found that for each quantum of radiation absorbed by a mercury atom, one molecule of hydrogen can, only under certain conditions, be converted into water vapour. Now if τ be the life period of the excited Hg atom and if T be the interval between two successive collisions between an excited Hg atom and a hydrogen molecule, it can be shown from ordinary method of chemical kinetics that the velocity of formation of water molecules will be given by $K\frac{T+\tau}{\tau}$. The interval between two successive collisions when the rate of formation of excited Hg atoms is kept constant is inversely proportional to the pressure of hydrogen gas.

Hence $$\frac{d[H_2O]}{dt} = K\frac{\tau}{T+\tau} = \frac{K.\tau.p}{A+\tau p},$$

where p=pressure of H_2, A is a constant which can be calculated from the kinetic theory of gases; and therefore the rate of change of velocity of water formation with the pressure of hydrogen gas can easily give us the value of τ. The average value comes out to be 10^{-7} seconds which is in agreement with the result of Wien derived from purely physical methods.

If, however, we take the simple case of photo-sensitisation of ozone by chlorine gas, we obtain an entirely different value of life period. Ozone decomposes into oxygen by absorption of ultraviolet light

2500 A. In presence of chlorine, this reaction takes place under visible light in the following way :

$$Cl + h\nu \longrightarrow Cl\ (N) + Cl\ (\text{metastable})$$

$$Cl\ (\text{metastable}) + O_3 \longrightarrow O_2 + O + Cl\ (\text{normal})$$

$$O + O_3 \longrightarrow 2\ O_2$$

$$Cl\ (N) + Cl(N) \longrightarrow Cl_2$$

The rates of decomposition of ozone under the influence of chlorine at different pressures of ozone have been determined by Weigert and recently by Boenhoeffer, and the life period of chlorine atom in the metastable state has been found to be 10^{-2} seconds. The active state of very short duration is called excited state while that of longer duration is called metastable state. Attempts have been made to explain the possibility of these two types of states from the standpoint of the selection principle, but definite and rational interpretation of each individual case is not yet available. I shall very briefly try to indicate to you the nature of such explanation. Fig. II shows the various levels of electron orbit in Hg atom and the origin of the more important lines of the spectrum of mercury. An outer electron in a mercury atom can by absorption of a quantum of monochromatic radiation 2536A, be pushed out to an outer 1^3p_1 orbit. From this orbit it can jump off to a still higher orbit 1^3S_1 by the further absorption of a quantum 4358A, or by collision with a second atom in which the kinetic energy is transformed into the potential energy of the higher quantum states. The electron in the 1^3S_1 orbit can come down to the 1^3p_0 state by emission of light of wave-length 4046A or by collision with another atom where the reverse transformation of quantum energy into kinetic energy can take place. This 1^3p_0 orbit of the electron represents a metastable state, because according to the Principle of Selection the electron can not jump directly from this orbit to the normal 1^1S_0 orbit. We might, therefore, expect a certain spurious stability for states such as 1^3p_0 from which a direct return of the electron to the normal state is impossible. The life of the

electron in 1^3p_1 orbit is of the order of 10^{-7} seconds while its life in 1^3p_0 orbit is of the order of 10^{-4} seconds.

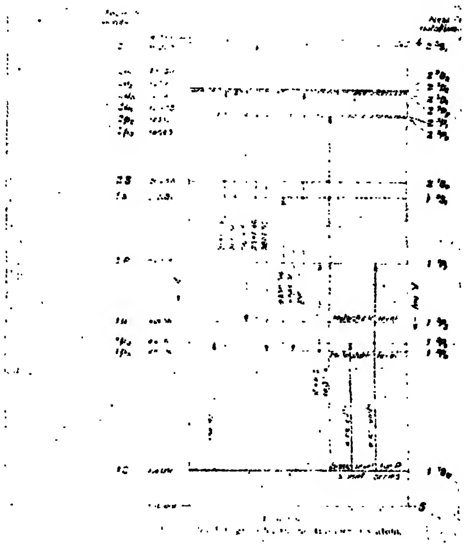


FIG. II.

From the cases of photosensitisation that I have cited, it should not be inferred that photo-chemical reactions are always as simple as in the examples given above. A simple reaction is an exception rather than the rule ; for example in the photosensitised transformation of allocinnamilidine acetic acid into normal form with iodine as sensitiser, depending on the nature of the solvent medium 1 quantum of light absorped by an iodine molecule, can transform between two to 1000 molecules of the allo-acid into the normal variety. Such facts can be explained by complicated chain reactions but such hypotheses from their very nature cannot rest on solid theoretical grounds.

It may not be without interest to apply the quantum theory to the photo-chemical reactions taking place in the human eye. Noddak finds that after complete adaptation in the dark, human eye can perceive light energy of 10^{-13} lumen seconds at 5400A. If we take nto consideration the reflection, absorption and scattering of light by the lens and the various septa of the eye, we come to the conclusion that the sensitive rods in the retina are each affected by 1 quantum

of green light. The human eye can, therefore, under ideal conditions, perceive one quantum of visible radiation.

I shall conclude by referring to some recent speculations on the interaction between units of radiation themselves to produce matter in the form of a proton and an electron. Compton has already postulated that a quantum of radiation has not only an energy $h\nu$ but has also got a momentum $\frac{h\nu}{c}$.

Let us suppose that when two quants of frequency ν collide with each other, a part E of the energy associated with the quants becomes latent by transformation into matter in the shape of a proton and an electron, and the remainder undergoes degradation into the heat motion of the proton and the electron;

$$E=2h\nu-2.\frac{3}{2}kT, \qquad \ldots \quad (1)$$

where T represents the temperature corresponding to the heat motion of proton or electron and k is the ordinary gas constant for a molecule.

Since energy and mass is connected by the relation

$$M=\frac{E}{c^2} \qquad \ldots \quad (2)$$

where c is the velocity of light, we obtain for the sum of the masses of the proton and the electron produced in the above process

$$M+m=\left[\ 2h\nu-3kT\ \right].\frac{1}{c^2} \qquad \ldots \quad (3)$$

The mass of an electron

$$m=\frac{2}{3}.\frac{e^2}{a} \qquad \ldots \quad (4)$$

where a is the radius of the electron and $e=1{,}59\ 10^{-20}$ E. M.U. Prof. W. C. McLewis assumes the radius of a light quantum, for which we may now substitute the name photon to equal that of an electron, while Ornstein and Burger conclude that a photon ought to have a radius equal to the wave-length of radiation associated with it.

If we assume that both these conclusions are correct in the particular case we arrive at the result

$$a=\lambda=\frac{c}{\nu} \qquad \ldots \quad (5)$$

and

$$m=\frac{2}{3}\cdot\frac{e^2.\nu}{c} \qquad \ldots \quad (6)$$

We assume further that when two photons bearing monochromatic radiation ν collide, the immediate result is that each photon is converted into a "Hohlraum" the volume and the total energy of the "Hohlraum" being the same as that of the proton which has been transformed. The space density of radiant energy in a photon

$$U=\frac{h\nu}{\frac{4}{3}.\pi.\lambda^3}, \qquad \ldots \quad (7)$$

where U is the energy density of total radiation in the "Hohlraum." According to Planck

$$U=\frac{48\pi.h}{c^3}.\left[\frac{kT}{h}\right]^4.\frac{\pi^4}{90} \qquad \ldots \quad (8)$$

where T is the temperature of the "Hohlraum." From (7) and (8)

$$kT=1{\cdot}1957\ h\nu \qquad \ldots \quad (9)$$

Identifying the temperature of the proton or the electron with the temperature of the "Hohlraum" in which they are produced, we obtain for

$$\frac{M+m}{m}$$

$$=\frac{6{\cdot}54\times10^{-27}.[2-3\times0{\cdot}1957]\nu.3\times3\times10^{10}}{3\times10^{10}\times3\times10^{10}\times2\times1{\cdot}59\times10^{-20}\times1{\cdot}59\times10^{-20}\times\nu} \qquad \ldots \quad (10)$$

$$=1828$$

The agreement with the experimental value may be purely fortuitous but is quite remarkable. It may well be that cosmic heat has its origin in the degradation of high frequency radiation into heat as

suggested above when the energy of such radiation is partially transformed into matter. The question remains open why radiation of wave-length $1{,}85 \times 10^{-13}$ cm. alone is capable of producing stable electrons.

The work of Millikan and Kolhörster, indicating the existence of cosmic rays of very high frequency, may give a touch of reality to this hypothesis.

1. J. Franck—Trans. Farad. Soc. *21* (1925).
2. Dymond—Z. Physik *34* (1925).
3. Turner—Phys. Rev. (2) *27* (1926).
4. Tinge and Gerke—J. Am. Chem. Soc. *48* (1926).
5. Berthoud—Trans. Faraday. Soc. *21* (1925).
6. J. C. Ghosh—J. Ind. Chem. Soc. (1925, 1926 & 1927).
7. J. C. Ghosh—Die Naturwissenschaften, May & June, 1927.

SECOND LECTURE

PHOTOSYNTHESIS IN PLANTS.

In my last lecture I discussed the nature of interaction between matter and radiation in those cases which are characterised by great simplicity. The region that I covered lay on the borderland between physics and chemistry. To-day I shall discuss the nature of the most fundamental, and at the same time, the most complicated photo-chemical reaction that we know of :—the syntheses of carbohydrates from CO_2,—and we shall traverse ground which lies between chemistry and plant-physiology.

It is well known that living plants build up complex organic compounds from CO_2 and moisture under the influence of light. This is the most fundamental photo-synthetic process we meet with in nature, and researches to elucidate the mechanism of this process have been carried out by innumerable investigators since the time of Ingenhousz, Senebier and de Saussure, who discovered at the end of 18th century that CO_2, moisture, and light are essential for photo-synthesis. To-day, I shall first speak to you about the attempts that have been made in the laboratory to obtain carbohydrates from CO_2 and H_2O under stimulus of light but without the aid of any plant pigment whatsoever. I shall then give you an account of the investigations on the influence of various physico-chemical factors, *e.g.*, intensity of illumination, concentration of carbon dioxide, temperature, etc., on the rate of carbon dioxide assimilation by living plants.

You are no doubt familiar with the hypothesis of Bayer according to which CO_2 is in the first instance reduced to formaldehyde, which undergoes condensation to form the hexoses and other complex bodies. The greatest obstacle to the acceptance of this hypothesis, that formaldehyde is a poison for plants, has been removed by the recent investigation of Sir Jagadish Bose who has shown that, in very minute doses, formaldehyde stimulates CO_2 assimilation. If the velocities of formation of formaldehyde and its conversion into hexoses are of the same order of magnitude, then it is clear that the concentration of formaldehyde at any time in the juices of the plant leaves may be exceedingly minute.

A direct reduction of CO_2 in aqueous solution to formaldehyde was accomplished by Fenton in 1907 by the use of metallic magnesium without the aid of any kind of light. That a similar rôle should be ascribed to the magnesium which constitutes part of the chlorophyll molecule in the photo-synthetic process, there is no reason for believing. But Fenton's results are important in the sense that it demonstrates the possibility of a direct reduction of CO_2 to formaldehyde.

Usher and Priestly reported in 1911 that they were able to obtain tests of formaldehyde by exposing quartz tubes containing water and carbon dioxide only to ultraviolet light of quartz mercury lamp. It is not without interest, that when they used open dishes containing water and placed these immediately beneath the lamp, they also obtained tests for formaldehyde even when no carbon dioxide was bubbled through the water. This investigation was extended by Baly and his co-workers who in 1921 claimed that synthesis of various complex organic compounds had been achieved in the laboratory by the action of ultraviolet light of extremely short wave-length on aqueous solution of CO_2. Prof. Porter in 1925 repeated with extreme care these experiments. He found that in every run where the illuminated gas came in contact with rubber tubing, sealing wax, de-khotinsky cement, or stop-cock grease, small quantities of formaldehyde were obtained but the results were not quantitatively reproducible. When the entire apparatus was constructed of quartz-to-glass seals and conductivity water and carbon dioxide made from pure sodium bicarbonate were used, after 36 to 60 hours' illumination by quartz mercury lamp, not a trace of formaldehyde or reducing sugar was obtained.

He came to the conclusion that as the purity of materials was increased, the yield of formaldehyde decreased until in the complete absence of all impurity no trace of formaldehyde was found. Chemical tests for formaldehyde are so sensitive that one part formaldehyde in 16 million parts of water can be detected, and Porter maintained that formaldehyde in Priestly and Usher's work and in that of Baly was obtained by the decomposing action of ultraviolet light on minute particles of organic dust present in water or carbon dioxide.

Matters were in this state when in September, 1927, Baly and his numerous collaborators published a series of papers on this subject. They claim that when an insoluble inorganic powder capable of absorbing carbonic acid on its surface is suspended in water, through

which is maintained a stream of carbon dioxide and when the whole is exposed to ultraviolet light, complex organic compounds are photo-synthesised. These may be recovered by evaporation of the solution after removal of the insoluble powder. They recommend basic carbonates of aluminium, magnesium and zinc as the most suitable material when ultraviolet light is used. If however a coloured powder like nickel or cobalt carbonate is suspended in water, photo-synthesis can be effected under similar conditions even by visible light. These results are similar to those obtained by Moore and Webster in 1914 on photosynthesis in presence of colloidal ferric or uranium hydroxide. If these results of Baly are found to be accurate by repeated investigations carried out with great care, then a partial solution of the problem of photosynthetic production of carbohydrates in vitro has been achieved.

I shall not enter into the vexed question of the successive stages of photosynthesis of complex carbohydrates from CO_2 and H_2O, because such a discussion in the imperfect state of our present knowledge is futile. I shall only mention one incontrovertible fact which has been established by the researches of Willstätter carried out with a very large variety of plants. It is this,—the amount of O_2 evolved is always equivalent to the amount of CO_2 absorbed. It necessarily follows that a reaction of this type $nCO_2+nH_2O \longrightarrow (HCHO)_n + nO_2$ takes place in the first instance. I shall only confine myself to a discussion of the rate of carbon assimilation.

The first systematic investigation of the influence of physical factors on carbon assimilation by living plants was begun by Blackman in 1895 and continued in collaboration with colleagues and students for about 15 years. The principle of his method is that the assimilating plant material is contained not in a completely closed chamber but in one through which passes a constant stream of gas containing a known proportion of carbon dioxide. After leaving the plant chamber, the gas passes through tubes in which carbon dioxide is absorbed and so determined. If the total quantity of air passing through the chamber is known, the total quantity of CO_2 entering the chamber can be calculated and hence the rate of carbon assimilation can be ascertained. As a result of his investigations Blackman came to the conclusion that for a definite intensity of illumination, the rate of assimilation increases as the concentration of CO_2 increases up to a certain point, beyond which any further increase in CO_2

concentration has no effect on rate of assimilation. Again for a definite concentration of CO_2, the rate of assimilation increases proportional to the intensity of light but beyond a limiting intensity further increase in the strength of illumination does not produce any change in the rate of assimilation. Blackman offered the following qualitative explanation of his experimental observation, known as the "principle of limiting factors." Suppose a green leaf is exposed to such illumination as will provide energy to reduce 5 c.c. of CO_2 per hour. If the concentration of CO_2 outside the leaf is such that only 1 c.c. of CO_2 is assimilated per hour, the energy provided by the light is sufficient for the whole of the CO_2 to be utilised in photosynthesis. If the concentration of CO_2 be now so raised that 2 c.c. of CO_2 is assimilated per hour, the energy of light is still more than sufficient to allow the utilisation of this quantity of CO_2. Now suppose the concentration of CO_2 is so raised that 5 c.c. of the gas can diffuse into the leaf in an hour, the energy supplied is now just sufficient to allow the assimilation of the CO_2. If the concentration of CO_2 is raised still further, increase in the rate of assimilation cannot take place because the energy supplied is only capable of decomposing 5 c.c. of CO_2 per hour. The light intensity is now the limiting factor in photosynthesis. Similarly when CO_2 concentration is kept fixed, and light intensity varied, the concentration of CO_2, beyond certain light strength, becomes the limiting factor in photosynthesis.

The experimental observations of Blackman have not been confirmed by later investigators. There can be little doubt that the continuous current method is the most reliable method for measuring photosynthesis. But the fact must not be overlooked that when excised leaves are used, the rate of photosynthesis is quite different from that obtaining in normal plant leaves. In excised leaves specially, the factor of translocation of materials formed in the leaves to other parts of plants has been eliminated. Later investigators therefore have concentrated their attention to aquatic plants, *e.g.*, Bose has carried out experiments with the aquatic plant Hydrilla, Warburg with Chlorella and Harder with the aquatic plant Fontinalis. All of them have come to the conclusion, that the curve representing the variation of photosynthetic activity with increase in CO_2 concentration, keeping light intensity constant, is a steep one at the beginning, has a turning point and then tends to reach a limit but this

limiting curve is not abruptly horizontal. The same relation is observed when CO_2 concentration is kept constant but light intensity varies. Of course, it should be remembered that in carrying out experiments concentration of CO_2 should not be raised so high as to stimulate the guard cells of the stomata to close the passage for CO_2 diffusion nor should the intensity be so increased as to stimulate the chloroplasts to huddle together and expose the least pigment surface to illumination.

In the short time at my disposal it will not be possible for me to describe in detail the methods of experiments and the apparatus devised by these investigators. In the Bose apparatus, the bubbles escaping from the plant are collected under an oil valve. When the gas attains a certain pressure, it lifts the oil valve and escapes. The apparatus is so devised that successive escapes represent equal volumes of gas. With this apparatus is associated an electric device which records by a dot on the drum of a clock each time the gas passes the valve. The apparatus is very ingenious and is very well suited for demonstrating the relative rates of photosynthesis under different conditions. Many plant physiologists are however of opinion that the amount of oxygen contained in the escaping bubbles varies with the rate of photosynthesis, and as the instrument does not measure the volume of oxygen emitted but the total gas, the measurements are only approximate.

The method of Harder does not suffer from this defect for he measures directly, by chemical analysis, the oxygen content of weak $KHCO_3$ solution in a closed glass vessel where photosynthesis by aquatic plants is proceeding. Winkler's method with suitable modifications has been used.

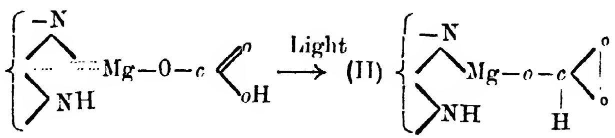
These recent results have been summarised by Boysen Jensen thus :—" The curve representing the influence of light on carbon dioxide assimilation, indicates that at the beginning where light intensity is small, light is the limiting factor, and in the last part, another factor, *e.g.*, CO_2 supply is limiting. Between these two parts there is a third part of considerable extent where the carbon dioxide assimilation is neither constant, nor proportional to intensity of light. Here the different factors interact." In what follows, I shall develop quantitative expressions for the rate of photosynthesis from the standpoint of modern chemical theories of chemical kinetics in heterogenous systems.

We shall take as our basis for quantitative treatment the hypothesis advanced by Willstätter and Stoll, on the mechanism of photosynthesis by plants. They believe that the following processes represent the successive stages in the assimilation of carbon dioxide :—

1. Carbonic acid is added to the magnesium complex of the chlorophyll and a carbonic acid derivative thus becomes a component of the pigment.

2. Light is absorbed by the chlorophyll carbonic acid complex, resulting in a molecular rearrangement in which the energy of radiation is locked up.

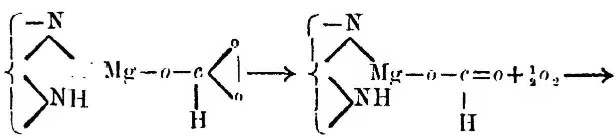
(I)



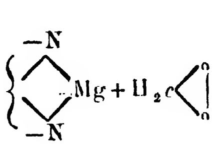
The chlorophyll carbonic acid complex (I) is transformed into peroxide (II).

3. The peroxide structure thus obtained is broken up by an enzyme of the katalase type, resulting in the regeneration of chlorophyll, evolution of oxygen and production of formaldehyde.

(III)



(IV)



We shall assume that N molecules of chlorophyll are originally present in unit area of the plastid surface, and that the chlorophyll molecules are uniformly distributed over this area. Now if the plastid is bathed in an aqueous solution of CO_2, then the number of chlorophyll molecules present as such per unit area will no longer be N but a fraction of N, because some of the chlorophyll molecules will form carbonic acid complex.

Again if this plastid surface is exposed to light, there should again occur a transformation of some of the chlorophyll carbonic acid complex molecules into molecules of peroxide. Now let us imagine that when the concentration of carbon dioxide is maintained constant at C, and the light intensity at I, per unit plastid area, the number of peroxide molecules at any time after a stationary condition has been reached be $N\alpha_1$, the number of addition complex of carbon dioxide be $N\alpha_2$ and the number of free chlorophyll molecules is necessarily $N \times (1-\alpha_1-\alpha_2)$.

Consider next, the rate of condensation of molecules of carbon dioxide on the plastid surface. From the theories of absorption, this rate will be proportional to the concentration of carbon dioxide in the aqueous phase and to the number of active centres of absorption per unit area.

Therefore rate of condensation of $CO_2 = K_1 N(1-\alpha_1-\alpha_2)C$. Again the rate of reverse transference of carbon dioxide from the surface to the aqueous phase is proportional to the number of chlorophyll carbon dioxide additive complex per unit area, *i.e.*,

Rate of transference to solution $= K_2 N\alpha_2$

The resultant rate of condensation of CO^2. when stationary condition has been reached is therefore

$$K_1 CN(1-\alpha_1-\alpha_2) - K_2 N\alpha_2 \qquad \dots \quad (1)$$

The rate of transformation of the chlorophyll carbonic acid additive complex into the peroxide under the influence of light is proportional to the intensity of illumination and to the number of additive complex molecules present per unit area, and hence

$$K_3 I . N\alpha_2 \qquad \dots \quad (2)$$

The rate of decomposition of peroxide molecules to reform chlorophyll, with the production of formaldehyde and oxygen is proportional to the number of peroxide molecules per unit area and the concentration of the ferment Katalase, and hence

$$= K'_4 N . \alpha_1 \text{ [Katalase]}.$$

Assuming that the ferment activity remains constant in the plastid bodies at a constant temperature, we obtain for the rate of evolution of oxygen the expression

$$= K_4 N\alpha_1 \qquad \dots \quad (3)$$

Now when, after a certain period of induction, depending on the nature of experimental conditions, a stationary state has been reached, we shall find that (*a*) the rate of condensation of carbon dioxide to form an additive complex with chlorophyll, (*b*) the rate of transformation of this complex to form peroxide, (*c*) the rate of regeneration of chlorophyll by decomposition of peroxide and (*d*) the rate of evolution of oxygen should all be identical. Thus for a constant value of photosynthetic activity, the following equations will hold good :

$$V \text{ (Velocity of carbon dioxide assimilation)} \quad \ldots \quad (4)$$

$$= K_1N(1-a_1-a_2)C - K_2Na_2 = K_3INa_2 \quad \ldots \quad (5)$$

$$= K_4Na_1 \quad \ldots \quad (6)$$

From (5) and (6) $a_1 = \frac{K_3}{K_4} . I\, a_2$;

from (4) and (5) substituting the value of a_1 in (5)

$$K_1N \left[1 - a_2 - \frac{K_3}{K_4}.I.a_2 \right] C - K_2Na_2 = K_3NI\, a_2$$

$$\text{or } a_2 = \frac{K_1C}{K_3I + K_2 + CK_1 + K_3/K_4 \cdot I.C.K_1} \quad \ldots \quad (7)$$

$$\text{or } V = K_3NIa_2 = \frac{K_3NI.\ K_1C}{K_3I + K_2 + CK_1 + \frac{K_3}{K_4}.I.CK_1} . \quad \ldots \quad (8)$$

$$\text{or } \frac{1}{V} = \frac{1}{NK_1C} + \frac{1}{K_3NI} + \frac{K_2}{K_3K_1NI.C.} + \frac{1}{K_4N} \quad \ldots \quad (9)$$

In the following pages, we shall discuss how far the validity of equations (8) and (9) is borne out by the experimental results of Harder on Fontinalis.

Table 1 contains a summary of Harder's experimental data on velocity of photosynthesis under varying conditions of experiment.

TABLE 1.

Intensity in Metre Candles.

$KHCO_3$	167MK	667MK	2000MK	6000MK	18000MK
·01%	·12	·41	·75, ·66* } ·71	·90	1·06
·04%	...	·91	2·24, 1·6* } 1·92	·3·45	4·7
·16%	...	1·1	3·45, 2·9* } 3·17	6·4	11·35
·32%	...	1·23	4·7, 3·1* } 3·9	8·6	15·2

In the 4th vertical column of Table 1 are given two sets of values of which the mean has been taken as the observed experimental data. The values marked with asterisk (*) are obtained by interpolation from Table 2 in Harder's paper (Jahrbuch f. Wiss. Botanik *60*, page 542) where the velocities of carbon dioxide assimilation at constant intensity of 2000MK, but with varying concentrations of $KHCO_3$ are recorded. The values in the 4th column not marked with asterisk are to be found in Table 5 of the same paper by Harder.

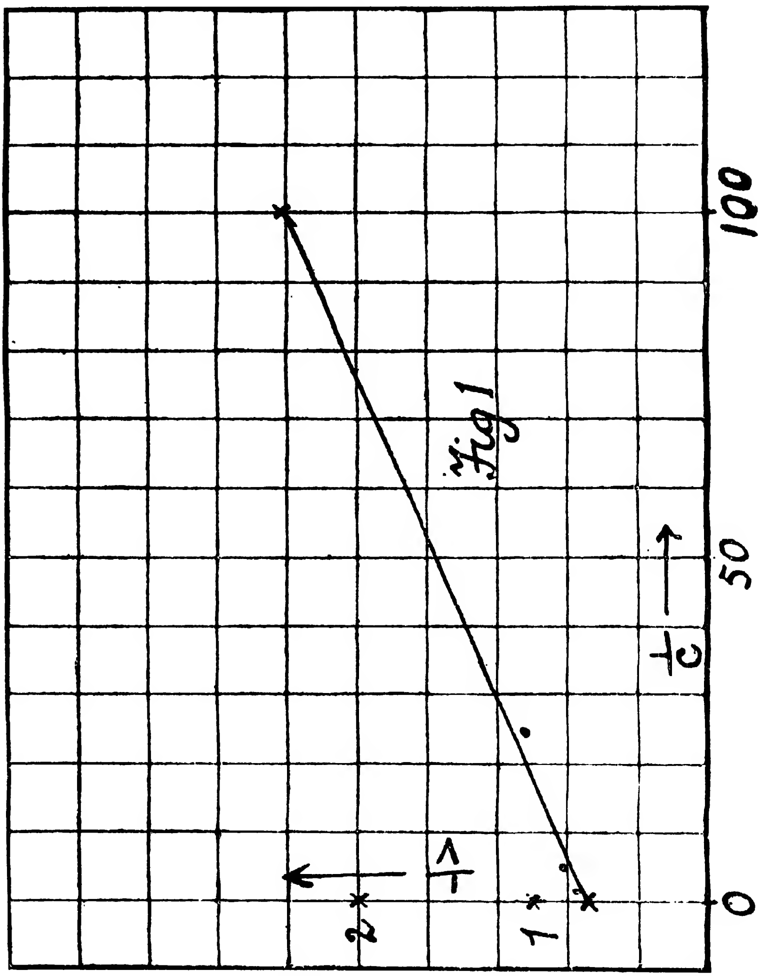
Variation of velocity with concentration of carbon-dioxide-light intensity remaining constant. From equation (9) it follows, that for a constant intensity of light I

$$\frac{d\left[\frac{1}{V}\right]}{d\left[\frac{1}{C}\right]}=\frac{1}{NK_1}+\frac{K_2}{K_3K_1NI} \quad \ldots \quad (10)$$

That is, if $\frac{1}{V}$ is plotted as ordinate against $\frac{1}{C}$ as abscissa, we should get a straight line, and the value of the

$$\text{slope}=\frac{1}{NK_1}+\frac{K_2}{K_3K_1NI}$$

In Fig. 1 are plotted Harder's values of $\frac{1}{V}$ against $\frac{1}{C}$, for constant light intensity of 667 MK, and it will be seen that the curve is a



straight line within the limits of experimental error. The intercept on the ordinate represents the value of $\frac{1}{V}$, when $\frac{1}{C} = 0$ or when $C = \infty$. From equation (9) this intercept is given by

$$\left[\frac{1}{V}\right]_{C=\infty} = \frac{1}{K_3NI} + \frac{1}{K_4N}$$

$$= 0{\cdot}70 \text{ (by extrapolation) when } I = 667\text{MK}. \qquad \ldots \quad (11)$$

In Table 2, at the bottom are given for each constant value of light intensity extrapolated values of $\frac{1}{V}$ for $C=\infty$. The best straight line passing through points plotted with $\frac{1}{V}$ as ordinate and $\frac{1}{C}$ as abscissa has been taken for obtaining the value of these intercepts. The calculated values in this table have been obtained from the equation,

$$\frac{1}{V} = \left[\frac{1}{V}\right]_{C=\infty} + \text{Slope} \times \frac{1}{C} \quad \dots \quad (12)$$

which is easily obtained from equations (10) and (11). The value of the slope in (12) can be calculated from one known value of $\frac{1}{V}$ at a known value of $\frac{1}{C}$, since $\left[\frac{1}{V}\right]_{C=\infty}$ is already known from extrapolation. Once the value of the slope, for a definite intensity of illumination is determined, all the data are available for finding out the value of $\frac{1}{V}$ for any value of $\frac{1}{C}$ at the same constant light intensity.

TABLE 2.

Velocity of Photosynthesis.

$KHCO_3$	667MK	2000MK	6000MK	18000MK
·01%	·41 obs.	·71 obs.	·90 obs.	1·06 obs.
·04%	·88 calc. ·91 obs.	1·89 calc. 1·92 obs.	2·93 calc. 3·45 obs.	3·8* calc. 4·7 obs.
·16%	1·1 calc. 1·1 obs.	3·18 calc. 3·17 obs.	6·7 calc 6·4 obs.	10·7 calc. 11·3 obs.
·32%	1·23 calc. 1·23 obs.	3·6 calc. 3·9 obs.	8·6 calc. 8·6 obs.	15·3 calc. 15·2 obs,
$C=\infty$,	V =1·43	4·13	11·8	27
$\left[\frac{1}{V}\right]_{C=\infty}$	·700	·242	·085	·037

It will be seen that with the exception of the data marked with asterisk, the calculated values of photosynthetic velocity agree with the experimentally observed values within the limits of experimental error. Equation 11,

$$\left[\frac{1}{V}\right]_{C=\infty}=\frac{1}{K_3NI}+\frac{1}{K_4N}$$

indicates that the reciprocal of velocity for infinite concentration of carbon dioxide is a function of the light intensity. Thus

$$\frac{d\left[\frac{1}{V}\right]_{C=\infty}}{d\left[\frac{1}{I}\right]}=\frac{1}{K_3N} \quad \ldots \quad (13)$$

That is, if $\left[\frac{1}{V}\right]_{C=\infty}$ is plotted as ordinate against $\frac{1}{I}$ as abscissa, a straight line should be obtained of which the slope is given by $\frac{1}{K_3N}$ and the intercept on the ordinate by $\frac{1}{K_4N}$. This has been done in curve I, Fig. 3, where the value of $\left[\frac{1}{V}\right]_{C=\infty}$ obtained by extrapolation, and recorded at the bottom of Table 2 have been plotted against the reciprocal of the corresponding light intensities. The curve as will be seen is a very good straight line, the value of the intercept being ·013.

Hence $0{\cdot}13=\frac{1}{K_4N}$ or $K_4N=77$.

It follows from equation (9) that K_4N is the limiting value of V, when the concentration of carbon dioxide and the intensity of illumination are both infinitely great or

$$[V]_{\substack{C=\infty\\ I=\infty}}=77$$

Variation of velocity of carbon assimilation with intensity of illumination, concentration of carbon dioxide remaining constant.

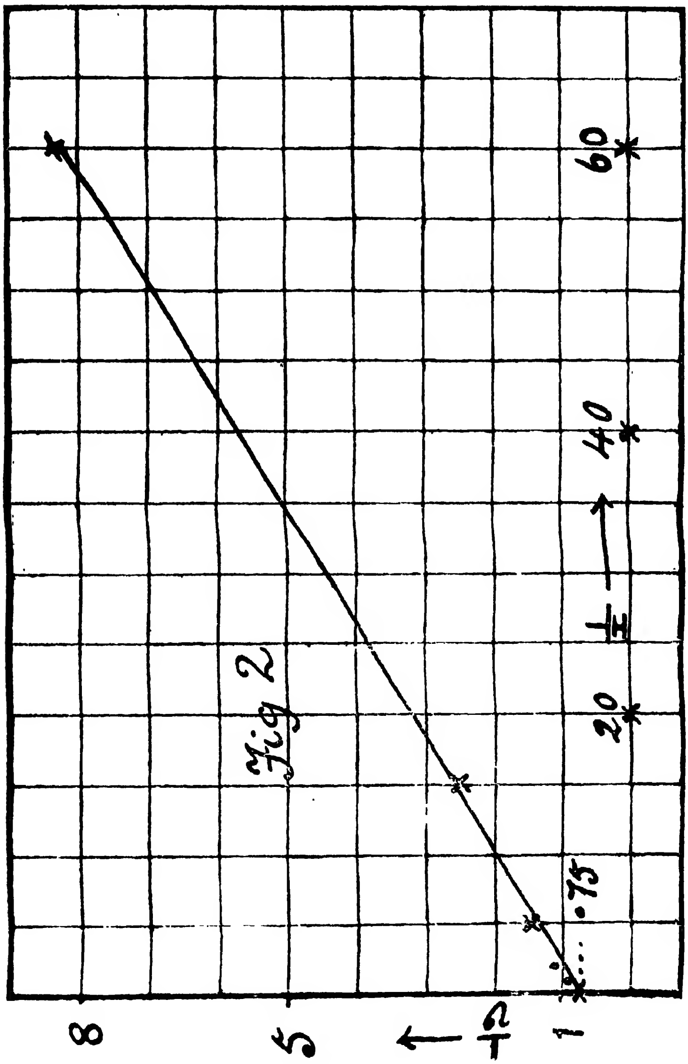
From equation (9) it follows that for a constant value of C

$$\frac{d\left[\frac{1}{V}\right]}{d\left[\frac{1}{I}\right]}=\frac{K_2}{K_3K_1NC}+\frac{1}{K_3N} \quad \ldots \quad (14)$$

That is, if for a constant value of carbon dioxide concentration, we plot $\frac{I}{V}$ as ordinate against $\frac{1}{I}$ as abscissa we should get a straight line, the slope of which will be given by

$$\frac{K_2}{K_3K_1NC}+\frac{1}{K_3N} \quad \ldots \quad (15)$$

In Fig. 2, are plotted Harder's values of $\frac{I}{V}$ as ordinate against $\frac{1}{I}$



as abscissa for constant concentration of $KHCO_3$ (·01%) and it will be seen that the curve is a good straight line. The intercept on the ordinate gives the value of $\left[\frac{1}{V}\right]_{I=\infty}$ =0·75, when C=·01%.

In Table 3, at the last vertical column are given the extrapolated values of $\left[\frac{1}{V}\right]_{I=\infty}$ for the different concertrations of $KHCO_3$. They have been obtained according to the method just indicated. The calculated values in Table 3, have been obtained by following the same procedure that was outlined before, that is, by determining the slope from the equation

$$\left[\frac{1}{V}\right]=\left[\frac{1}{V}\right]_{I=\infty}+\text{slope}.\frac{1}{I}$$

from a given value of V at a known values of I, concentration of $KHCO_3$ remaining constant. It will be noticed that the calculated values agree with the observed values within the limits of experimental error.

TABLE 3.

Velocity of Carbon Assimilation.

$KHCO_3$	167MK	667MK	2000MK	6000MK	18000MK	∞Mk	$\left[\frac{1}{V}\right]_{I=\infty}$
·01%	·12 obs.	·41 obs. ·38 calc.	·71 obs. ·72 calc.	·90 obs. 1·04 calc.	1·06 obs. 1·22 calc.	1·33	0·75
·04%		·91 obs.	1·92 obs. 2·0 calc.	3·45 obs. 3·4 calc.	4·7 obs. 4·3 calc.	5·08	0·197
·16%		1·1 obs.	3·17 obs. 2·92 calc.	6·4 obs. 6·54 calc.	11·3 obs. 11·1 calc	17·2	0·058
·32%		1·23 obs.	3·9 obs. 3·4 calc.	8·6 obs. 8·3 calc.	15·2 obs. 15·6 calc.	28·5	0·035

From equation (16)

$$\left[\frac{1}{V}\right]_{I=\infty}=\frac{1}{NK_1C}+\frac{1}{K_4N}$$

or
$$\frac{d\left[\frac{1}{V}\right]_{I=\infty}}{d\left[\frac{1}{C}\right]}=\frac{1}{NK_1} \quad \cdots \quad (17)$$

Hence if $\left[\frac{1}{V}\right]_{I=\infty}$ that is, the values given in the last vertical column of Table 3, are plotted as ordinate against corresponding values of $\frac{1}{C}$ as abscissa, we should get a straight line whose slope is

given by $\frac{1}{NK_1}$ and the intercept of this curve on the ordinate will give the reciprocal of the velocity when both I and C are infinitely large. Curve II, Fig. 3 represents this graph and it will be seen that

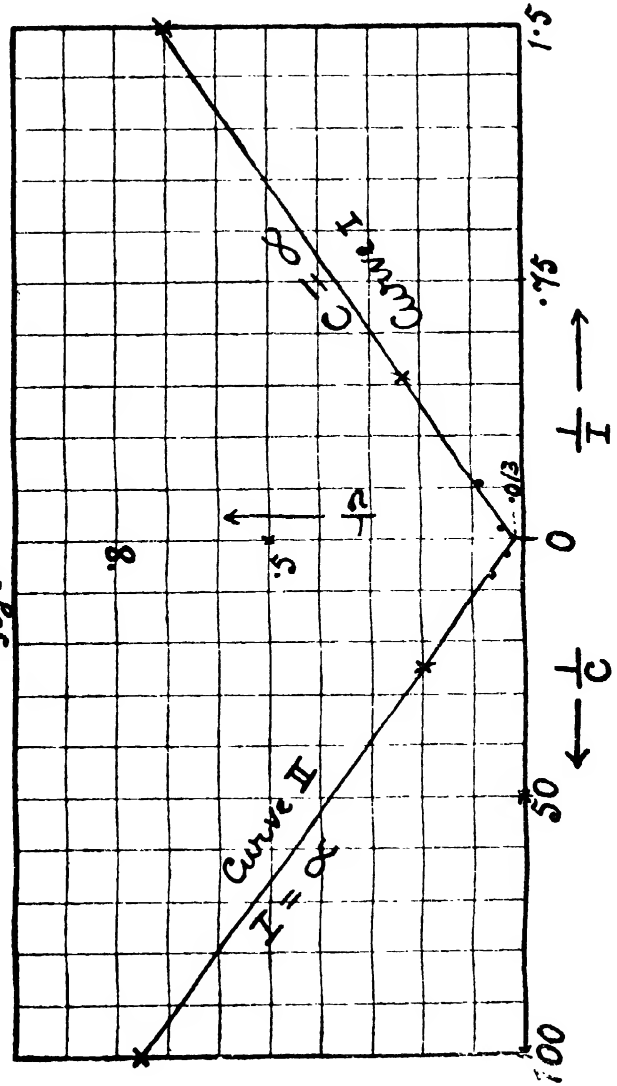


Fig 3

it is a very good straight line and that the value of the intercept on

the ordinate is the same as given by curve I. Equation (9) demands such identity for at infinite values of I and C,

$$\left[\frac{1}{V}\right]_{\substack{I=\infty\\C=\infty}} = \frac{1}{K_4N} = 0{\cdot}013 \quad \ldots \quad (18)$$

or V=77, when I and C are both ∞.

The fact that $\left[\frac{1}{V}\right]_{I=\infty}$ when plotted against $\frac{1}{C}$ gives the same extrapolated value of $\left[\frac{1}{V}\right]_{\substack{I=\infty\\C=\infty}}$ as that obtained by plotting $\left[\frac{1}{V}\right]_{C=\infty}$ against $\frac{1}{I}$ may be taken as a very strong evidence of the validity of the theory advanced in this paper.

Evaluation of the constants Nk_1, Nk_2, Nk_3, Nk_4.

We have seen that $\left[\frac{1}{V}\right]_{\substack{I=\infty\\C=\infty}} = \frac{1}{Nk_4}$

and hence $Nk_4=77$ ... (19)

$$\text{Again} \quad \frac{d\left[\frac{1}{V}\right]}{d\left[\frac{1}{C}\right]} = \frac{1}{Nk_1} \quad \text{when } I=\infty \quad \ldots \quad (17)$$

From Table 3, $\left[\frac{1}{V}\right]_{I=\infty}$ changes from 0·75 to 0·013

when $\frac{1}{C}$ changes from 100 $\left(\text{or } \frac{1}{\cdot 01}\right)$ to 0 $\left(\text{or } \frac{1}{C\,\infty}\right)$

$$\therefore \text{ The slope} = \frac{0{\cdot}75-0{\cdot}013}{100} = 0{\cdot}00737 = \frac{1}{Nk_1} \quad \ldots \quad (20)$$

or $Nk_1 = 135{\cdot}7$, when 0·01% $KHCO_3$ is taken as the unit of concentration.

$$\text{Again} \quad \frac{d\left[\frac{1}{V}\right]}{d\left[\frac{1}{I}\right]} = \frac{1}{k_3 N} \quad \text{when } c = \infty \qquad \ldots \quad (13)$$

Taking 100 Mk as the unit of I, we obtain from Table 2, that

$\left[\frac{1}{V}\right]_{C=\infty}$ changes from 0·70 to 0·013 as I changes from 0·667 to ∞

$$\text{Therefore the slope} = \frac{0{\cdot}70 - 0{\cdot}013}{\frac{1}{0{\cdot}667}} = 0{\cdot}458 = \frac{1}{k_3 N}$$

$$\text{or } k_3 N = 2{\cdot}18 \qquad \ldots \quad (21)$$

The value of $k_2 N$ can be ascertained in two possible ways. Thus when c is constant

$$\frac{d\left[\frac{1}{V}\right]}{d\left[\frac{1}{I}\right]} = \frac{1}{k_3 N} + \frac{k_2}{k_3 k_1 NC} = \frac{1}{k_3 N} + \frac{k_2 N}{k_3 N . k_1 N . C} \qquad \ldots \quad (14)$$

From the data in the second horizontal column in Table 3, we find that, when $c = 0{\cdot}01\%$

$$\frac{d\left[\frac{1}{V}\right]}{d\left[\frac{1}{I}\right]} = \frac{\frac{1}{0{\cdot}12} - 0{\cdot}75}{\frac{1}{0{\cdot}167}} = 1{\cdot}260$$

But $\frac{1}{k_3 N}$ from (21) $= 0{\cdot}458$ and $\frac{1}{k_1 N}$ from (20) $= 0{\cdot}00737$

$$\text{Hence } k_2 N = \frac{0{\cdot}802 \times 0{\cdot}01}{0{\cdot}458 \times 0{\cdot}00737} = 2{\cdot}35 \qquad \ldots \quad (22)$$

Again when I is kept constant

$$\frac{d\left[\frac{1}{V}\right]}{d\left[\frac{1}{C}\right]}=\frac{1}{Nk_1}+\frac{k_2}{k_3k_1NI}=\frac{1}{Nk_1}+\frac{k_2N}{k_3N.k_1N.I} \quad \ldots \quad (10)$$

From the data in the second vertical column in Table 2, we find that when $I=0{\cdot}667$

$$\frac{d\left[\frac{1}{V}\right]}{d\left[\frac{1}{C}\right]}=\frac{\frac{1}{0{\cdot}41}-0{\cdot}70}{100}=0{\cdot}0174$$

But $\frac{1}{Nk_1}$ from $(20)=0{\cdot}00737$

$$\therefore k_2N=\frac{0{\cdot}01003\times0{\cdot}667}{0{\cdot}458\times0{\cdot}00737}=2 \quad \ldots \quad (23)$$

The agreement between the two values of k_2N (2·35 and 2) are as good as can reasonably be expected.

Warburg's Experimental Observations.

(Biochem. Zeitschrift *100*, p. 253.)

The results obtained by O. Warburg on the rate of photosynthesis by Chlorella under different concentrations of carbon dioxide and light intensities, can also be adduced as experimental evidence in favour of the kinetics of photosynthesis developed above.

Table 4 contains Warburg's results on assimilation rates as dependent upon concentration of carbon dioxide

TABLE 4.

Concentration of $CO_2 \times 10^6$	·53	1·0	2·6	5·3	9·8	23	43	91
Assimilation per hour } obs.	29	47	120	144	178	202	214	242
Assimilation calculated }	—	49	98*	142	177	212	228	240
	V=250 for C=∞							

The value of 250 as the rate of assimilation for infinite conc. of CO_2 was obtained by extrapolation of the curve $\frac{1}{V}$ against $\frac{1}{C}$, and the calculated values in the third horizontal column have been obtained according to the method already described. Excepting the data marked *, the agreement between observed and calculated values is fairly good.

The Table 5 contains Warburg's results on assimilation under different conditions of light intensity.

TABLE 5.

Intensity of light.	1	2	4	7·1	16	∞
Assimilation per hour obs. ...	15	85	135	172	220	—
Assimilation calculated ...	—	79	125	167	222	300

The assimilation rate for infinite light intensity has been obtained by extrapolation from the curve of $\frac{1}{V}$ plotted against $\frac{1}{I}$ which has been found to be a very good straight line

It will thus be seen that Warburg's experiments, within the limited range of their applicability, supply strong evidence in favour of the theory outlined here.

Influence of Temperature on the Rate of Photosynthesis.

Harder's experiments were all carried out at a constant temperature of 21°·8. It is not possible therefore to determine how the constants K_1, K_2, K_3 and K_4 would vary with temperature. From general principles of chemical kinetics, it may however be surmised, that if the molecular re-arrangement of the carbonic acid chlorophyll complex under the stimulus of light obeys even approximately Einstein's law of photo-chemical equivalence, K_4 will be practically independent of temperature. Again K_1 which indicates relation between the rate of condensation of CO_2 on a surface and the concentration of CO_2, can be expected to increase with temperature at the same rate as the velocity of the molecule. The other constants K_2 and K_4 are

complex exponential functions of temperature. Now from equation (9) the value of $\frac{1}{V}$ may become fairly independent of temperature when $\frac{1}{K_3 N I}$ becomes very large in comparison with the other terms. We may thus roughly say that for very small values of light intensity $\frac{1}{V}$, and hence also velocity of photosynthesis, is almost independent of temperature This has been found to be the case by Blackman and by Warburg. It appears however that complete elucidation of the complex problem of influence of temperature on rate of photosynthesis must await further experimental investigation.

THE UNIVERSITY,
DACCA,
INDIA.

On the Branchio-cephalic System of Blood-Vessels, together with a note on the Dorsal Aorta of a Common Indian Fresh-water Carp, 'Rohu,' *Labeo Rohita* (Günth.).

BY

PURNENDU SEN, M.Sc.

Research Scholar, Zoology Department,
University of Calcutta.

With 9 Plates.

CONTENTS.

INTRODUCTION.

Having made a thorough enquiry into the up-to-date literature on Ichthyology available in India I was unable to find a suitable description of the anatomy of any Indian fresh-water fish, and more particularly the vascular system in general. It was, therefore, suggested to me by Dr. B. K. Das, University Professor of Zoology,

University of Calcutta, that it might be worthwhile to work out systematically and in detail the anatomical features of a fairly known Indian Teleost with a view to throw more light on the structure and life-history of the Indian Fauna in general as well as to fill up the need of College curricula. Acting on his suggestions my attention was primarily drawn towards the vascular system of one of the common Indian Carps, 'Rohu,'—a specialised form whose general organisation simulates a typical condition amongst the fish tribe, and the results of my investigation with regard to the anterior blood vessels are presented in the following pages.

MATERIAL AND METHODS.

Fishes were usually obtained alive, and fresh dissections were made in the majority of cases, the exact course of the blood vessels being determined by the help of a binocular. In certain doubtful cases, however, the materials were injected with Carmine and preserved in a mixture of 4 per cent. Formalin and Glycerine (the strength of the latter being 10 per cent. in the mixture and, this, besides possessing other qualities, renders the tissues soft for section cutting) for about a week till they were sufficiently hardened to allow a successful dissection. Wherever bones were encountered they were softened in Schridde's solution (=equal parts of HNO_3 and Formol and 9 parts of water) for at least 4 to 5 days after preservation, and following this process of decalcification the laborious task of delicate dissections was made considerably lighter.

Here I wish to express my great indebtedness to Dr. B. K. Das for his constant help, encouragement and kind criticisms.

THE HEART, ITS STRUCTURE, THE ARRANGEMENT OF THE VALVES, ETC.

Like the Teleostean heart in general the heart of the 'Rohu' consists of the auricle (Pl. I, Au.), the ventricle (Vent.), and the sinus venosus (S. V.) enclosed within a thin-walled bag, the pericardium, and is situated about 6·5 cms. behind the tip of the snout in a fish measuring 34·4 cms. in length. Anteriorly the pericardial wall is attached with the hind part of the bulbus aortæ, whilst posteriorly it is connected with the great veins entering the sinus venosus. Ventrally it is

Plate I

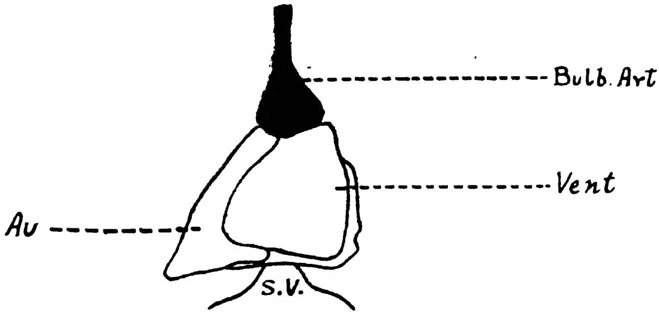


Fig. 1

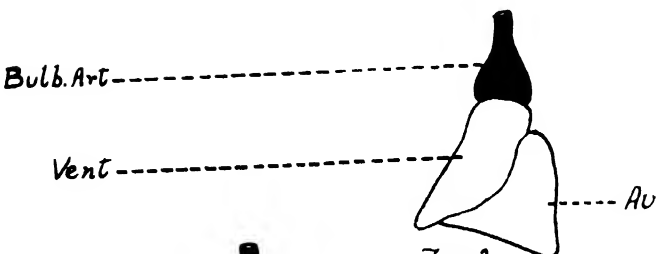


Fig. 2

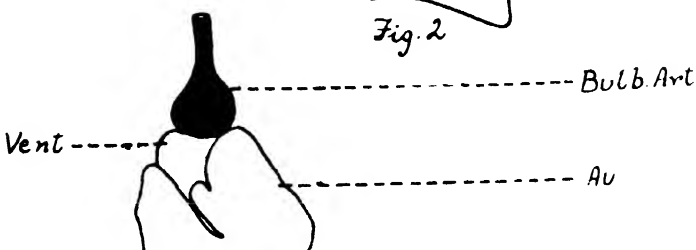


Fig. 3

bounded by the coracoids and the hyo-pectoralis muscles, and the latter meeting together in front give a triangular appearance to the whole of the cardiac region.

The sinus venosus is a thin-walled triangular sac receiving venous blood poured in by the two large Ducti Cuvieri, the Hepatic veins and the single Jugular vein. Anteriorly, however, it embraces the hinder wall of the auricle and the communication between the latter and the sinus is established by means of a hole,—the sino-auricular aperture, guarded by a couple of membranous valves (Pl. II, S. Au. V.) ; the inner wall of the sinus venosus is fairly smooth in character.

The auricle is a somewhat flattened chamber about 1·65 cms. in length. It has a broad base which is disposed transversely across the long axis of the head in a fish whose heart measured 2·22 cms. in its total length. It lies more towards the left side of the heart (Pl. I, Au.), and then curves round over the dorsal surface of the ventricle, reaching the posterior limit of the bulbus ; its margin presents a crenulated appearance. In sagittal section it is more or less like a right-angled triangle, and resembles that of the heart of Polyodon (Danforth,—8), being high behind and low in front. It is characterised by the presence of a large number of trabeculæ along its inner walls, which, uniting with one another, divide the chamber into several small inter-communicating blood-compartments. The auricle leads into the ventricle by means of an almost circular opening known as the auriculo-ventricular aperture guarded by valves (Pl. II, Au. Vt. V.), the aperture being itself placed anterior to and at a lower level than the sinu-auricular opening. It should be borne in mind, however, that the auricular wall is not so thick and muscular as the ventricle, and the copious development of the trabecular system apparently ensures a rapid contraction of the auricle and thus enhances the pumping of blood with great force towards the ventricle.

So far as I have examined the condition of the heart in Indian teleosts the question of valves presents very many interesting features. Here in 'Rohu' we find a couple of membranous valves guarding the sinu-auricular aperture. These valves are situated dorso-ventrally and one of them, *viz.*, the dorsal one, is continued forward as a frill up to the margin of the large auriculo-ventricular aperture (Pl. II, Fig. 2, S. Au. V.), having its ventral border attached to the ventral wall of the auricle. The number of valves at the auriculo-ventricular aperture is very variable. Normally there are four pocket valves (Au.

Vt. V.), of unequal size, and these are placed in pairs more or less, that is, two of the largest valves (forming a pair) lie ventral to the other pair, and the members of the same pair almost face against each other. These valves are quite prominent and can be easily made out on opening the ventricular cavity and looking at them end-on view, but in addition to these I have found as many as four small accessory pockets (Pl. II, Fig. 1, Accr. V.) lying at the angles subtended by any two of the larger valves and dorsal to them, so that in all it makes a total of eight valves (Pl. II, Fig. 3, 1-8)—a feature worthy of note and is so far not recorded in any other Teleostean fish. Some of these accessory valves, however, are crossed over by fine trabecular strands, and it may be said their number is subject to variation depending upon the age and size of the fish, in smaller fishes only two accessory valves being observed. Mention may also be made that the cardiac area enclosed by some of these valves presents more or less a fenestrated appearance due to its spongy character.

The ventricle is an elongated, highly muscular cylindrical structure, having a slight curvature towards its dorsal surface. It is about 1·62 cms. in length in the heart of the fish referred to above. Over its outer walls I found some traces of lymphoid tissue intermingled with fat, and it is supplied with blood by means of the Hypobranchial artery (Pls. IV & V, Hyb. Ar.). The wall of the ventricle is extremely thick and is chiefly composed of muscular fibres and thrown internally into ridges and folds ; the trabeculæ, though present, are fewer in number as compared with those of the auricle. The muscular folds leave a central cavity extending from the auriculo-ventricular aperture up to the opening into the bulbus.

In Teleost in general, as it is well-known, the conus arteriosus (Pl. II, Con. Art.) undergoes a great reduction, and, as such in 'Rohu,' we find that the conus is almost abortive, and is represented by a mere small passage lying between the ventricle and the bulbus aortæ and guarded by a pair of semilunar pocket-valves (Pl. II, Con. V.), the outer edges of which are drawn forward as fine strands over a short distance and inserted along the inner wall of the bulbus (like the ill-defined chordæ tendineæ of higher vertebrates). These valves are disposed along the lateral walls of the conus.

PLATE II

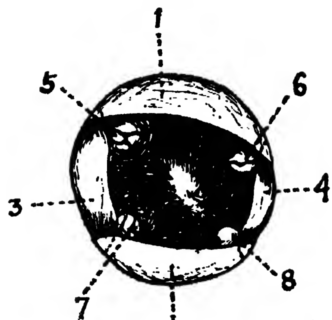


Fig. 3

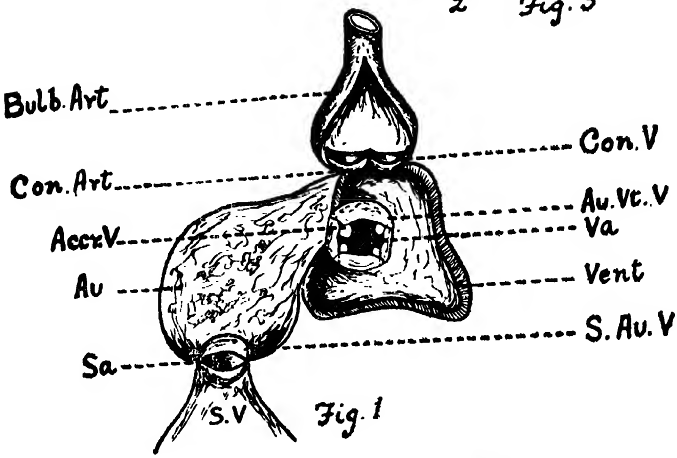


Fig. 1

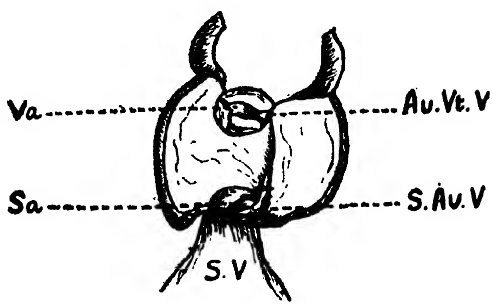


Fig. 2

BLOOD-CIRCULATION OF THE GILLS.

The Afferent System of Vessels.—The impure blood collected in the heart is pumped for purification to the gills by means of an elongated tabular vessel known as the Ventral Aorta (Pls. III, IV & V, V. Ao.) proceeding forward from the ventricle and lying along the under surface of the floor of the mouth-cavity. Its basal part ends in a small non-contractile dilatation called the bulbus aortæ (Pl. I, Bulb. Art.). The ventral aorta is about 3·6 cms. in length in the same specimen as mentioned above. At a level with the region of the first pair of gill arches the ventral aorta divides into two afferent branches (Pls III, & V, Br. Af_1) each entering into the corresponding gill arch. The second pair of afferent branchials (Br. Af_2) is given off at a distance of 3mm. from the first pair and supply the corresponding gills; and at a distance of 5·5 mm. behind the second pair is given off the third afferent branchial vessel (Br. Af_3) from either side of the Ventral Aorta and supply the gill of its side. The fourth pair of branchial afferent (Br. Af_4), however, lies nearer the heart, being 9 mm behind the third pair of afferents and 1·5 cms. in front of the bulbus. It may be mentioned that this last pair of branchial vessels does not arise from the lateral side of, and at right angles to, the long axis of the ventral aorta, as do the other three pairs of branchial afferent vessels, but at their origin they make an angle of about 30° with the latter and emerge slightly from thedorsal side of the same. They run backwards for some distance before they meet the corresponding branchial arches, and then they penetrate the ceratobranchials in order to supply the gills.

It is interesting to note that in a few instances certain small branches were seen to proceed from the muscles of the chin and to open at the point of bifurcation of the ventral aorta; this may be considered as an individual variation.

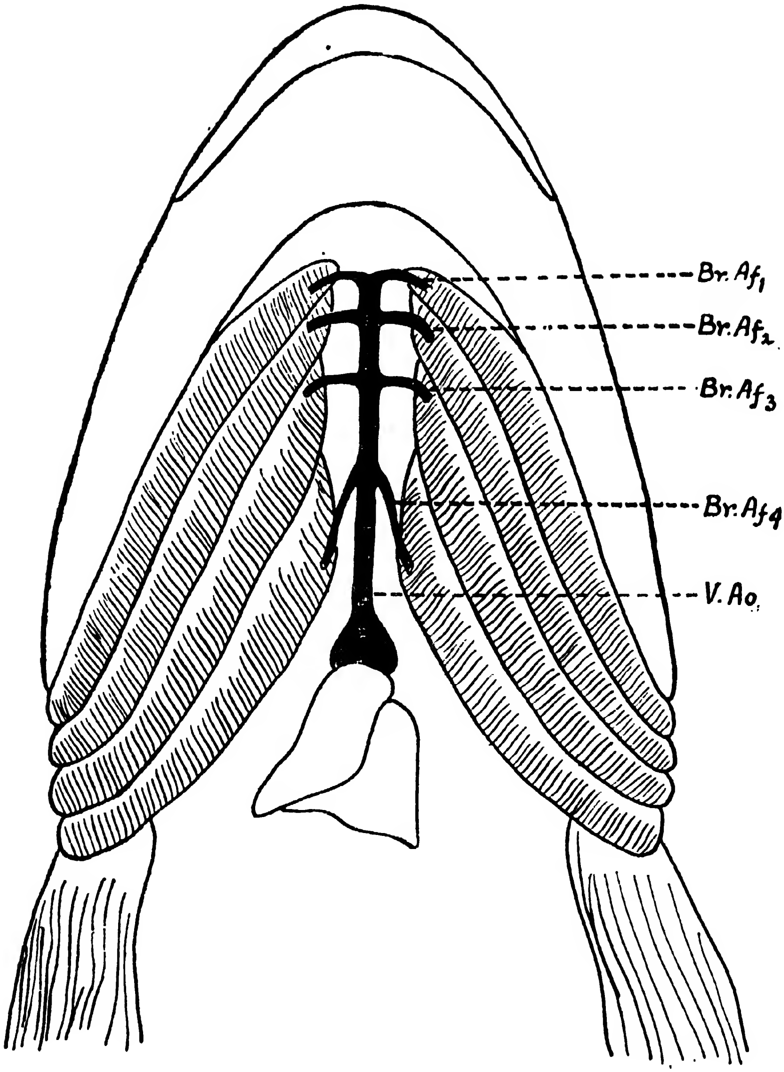
The Efferent System of Vessels.—After the blood is being purified at the gills the oxygenated blood is returned and sent to all parts of the body *viâ* the efferent branchial vessels (Pl. VI, Br. Ef_1-Ef_4) and the dorsal aorta (D. Ao.). There are four efferent branchial vessels which bring back blood from the corresponding gills. The first and the second efferents of either side at first run parallel to each other at a distance of nearly 2·6 mm. apart and then curving inwards unite to form the first pair of suprabranchial arteries (Sup. br. Ar_1),

which after traversing a distance of 4·5 mm. dorsally inwards unite together to form a single vessel, *viz.*, the anterior part of the dorsal aorta; the point at which the two anterior suprabranchials unite is about 4 cms. from the snout of the fish. The dorsal aorta, after pursuing a course of 1 cm. along the palate, catches up the posterior pair of suprabranchial arteries (Sup. br. Ar_2) at its ventral surface where the two, after running for nearly 4 mm., curve inwards towards the middle line and open together by means of a common aperture, the posterior pair of suprabranchials being formed by the union of the 3rd and 4th efferent branchial vessels which are similarly disposed as the first two pairs of efferents just mentioned above, and lie about 3 mm. apart from each other; the intervening space between the second and third efferent branchial vessels is about 4·5 mm.

Two very important arteries are given off from the first efferent branchial vessel, *viz.*, the external carotid (Ex. C.) and the internal carotid (In. C.). The external carotid artery arises from the dorsal side of the first efferent vessel at a distance of about 2·5 mm. from the junction of the first and the second efferent vessels. It passes upwards along the lateral side of the head and then turns round and ends over the region of the snout. During its course it gives off a mandibular branch (Md. Ar.) which runs along the outer surface of the lower jaw and supplies the latter with pure blood. A second branch goes to the muscles of the orbit (Opt. Ar.), and posteriorly a very long vessel arises from the external carotid which supplies the inner side of the operculum (Op. Ar.) and from the same another small artery is given off to the preopercular region (Prop. Ar.).

The internal carotid artery, however, arises from the anterior surface of the first efferent branchial vessel a little in front of the origin of the external carotid artery. About 3·5 mm. from the point of its origin the internal carotid artery divides into two main branches,—the outer one (Cr. Ar.) passing forward along the dorsolateral part of the mouth cavity and giving branches to the muscles of that region, whilst the inner one (Cerb. Ar.) forms the cerebral artery supplying the brain, and gives a small median branch which, anastomosing with its fellow of the opposite side, constitutes the circulus cephalicus. There is also a superficial vessel (Pal. Ar.) arising from the first efferent branchial about 1 mm. away from the point of origin of the internal carotid artery. It runs towards the mid-dorsal region of the palate which it supplies. Situated a short distance in front of the

PLATE III



origin of the carotids there lies a glandular body (Sp. Psd.) known as the Spiracular Pseudobranch (*i.e.*, the so-called hyoidean pseudobranch). It is supplied by means of a small vessel called the "Hyoidean" artery arising from the first efferent branchial vessel. The efferent branch (Opth. Ar.) emerging from the pseudobranch is known as the ophthalmic artery and supplies the choroid gland of the eye. At a distance of about 2 mm. from the junction of the first and second efferent branchials there is given out a small vessel from the latter which forms the auditory branch (Ot. Ar.), running at first almost parallel with the dorsal aorta and supplying the muscles of the otic region.

So far we have considered the mode of origin and the general disposition of the vessels connected with the branchae; but the exact mode of distribution of the finer branches of the branchial vessels and their circulation has not been discussed. I will, therefore, now proceed on with a brief account of the actual condition and the fate of the branches of the gill-vessels inside the gill filaments.

As already described the afferent branchial vessels arising from the ventral aorta pass outwards towards the gill arches; they run for some distance along the arches and on reaching the ceratobranchials each of these divides into two branches, supplying the hemibranchs. The blood vessels of the gill arches, as it is quite well-known, are lodged in a distinct groove along the arches, and the afferent branchial vessel (Pl. IX, Br. Af.) is situated ventral to the efferent vessel (Br. Ef.). As the afferent vessel goes along the arch it gives off a couple of fine twigs (Br. Af. fl.) carrying impure blood from its either side which traverse the inner edges of a pair of gill-filaments in which they capillarise as shown diagrammatically in the figure; the purified blood being collected and brought back by the corresponding efferent twigs (Br. Ef. fl.) which curve dorsally round over the afferent vessel and then ultimately open into the efferent branchial vessel. This type of arrangement of the branchial circulation seems to exist in the majority of fishes, as for instance McKenzie (Proc. Canad. Inst. 1883-84) has described a similar condition in the American Cat-fish,—Amiurus; and Danforth (1912) states that Allen (1907) has described a similar state of affairs in Polyodon and a like condition being also described by Agassiz and Vogt in connection with the anatomy of Salmon.

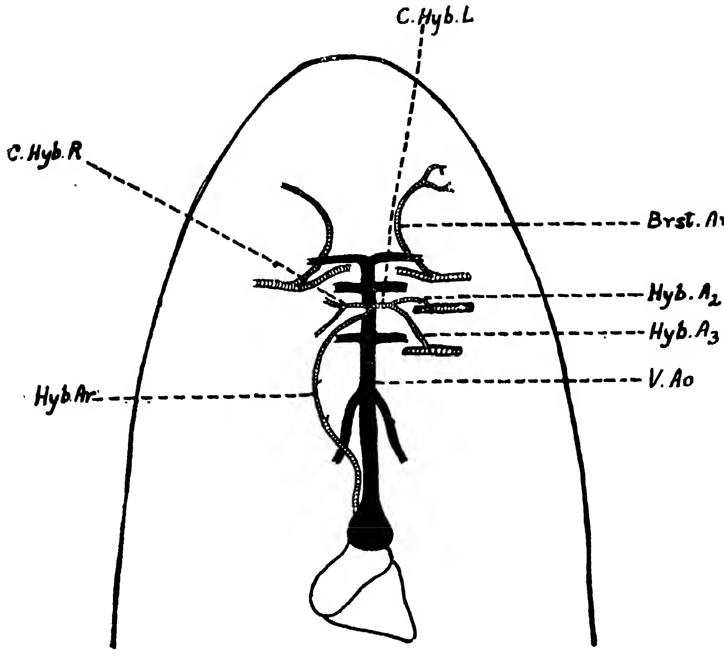
It is extremely interesting to note that the plan [1] of the efferent branchial system in 'Rohu' appears to be intermediate between the African Cat-fish, Melopterurus and the common European Carp, *Cyprinus carpio* and very nearly links with that of the European Loach, *Cobitis taenia.* This fact is of positive importance when one reflects upon the relationship between the Siluridae and the Cyprinidae.

In connection with the arterial system of this fish another important artery deserves special mention, *viz.*, the Hypobranchial (Pls. IV & V, Hyb. Ar.). The Hypobranchial artery gives out the coronary factor supplying the heart muscles as well as other small vessels supplying the gills. It lies ventral to the Ventral aorta, and is formed by the union of two common vessels (C. Hyb. R. and C. Hyb. L.), each 2·6 mm. in length, from either side of the branchial chamber, the junction being situated below the Ventral aorta in between the second and the third afferent branchial vessels at a distance of 2mm. and 3·8 mm. respectively from the proximal ends (*i.e.*, nearer their points of origin from the ventral aorta) of these blood-vessels. Each of these common vessels in its turn is formed by two factors, the anterior one (Hyb. A_2) emerging from the second efferent branchial vessel and the posterior one (Hyb. A_3) from the third efferent branchial. Several authors have studied the hypobranchial arteries in detail in other Teleosts, but amongst them notable ones are Parker (1900) and Silvester (1904). In 'Rohu' I observed certain minor variations as to the arrangement of the factors of the Hypobranchial of the two sides like those of Polyodon, but there was no divergence from the normal condition as has been incidentally recorded in a few fishes.

The median Hypobranchial artery (Hyb. Ar.) is about 2·38 cms. long in a fish whose length is 31·9 cms., and at first pursues a backward course alongside the muscles of the urohyal in the form of an arch directed away from the ventral aorta. It then gradually ascends upwards near the anterior region of the pericardium and finally divides itself into capillaries over the cardiac region. This sort of plan of the Hypobranchial artery is, however, met with in the American Cat-fish, Amiurus, but there the artery is formed by the factors derived from the third and the fourth efferent branchials; in 'Rohu' as we have

[1] According to Ridewood the condition of the efferent branchial system in 'Rohu' is a case of specialisation.

PLATE IV



just seen, there was no contribution towards the formation of this artery from the first or the fourth efferent vessel.

It is also interesting to observe that the first efferent branchial vessel, while still lying inside the gill-arch, gives out a small artery (Brst. Ar.) which at first runs forwards and then backwards along the inner surface of the operculum and ends into the region of the branchiostegals which it supplies.

DISPOSITION OF THE DORSAL AORTA AND ITS PRINCIPAL BRANCHES.

Lying just in front of the opening of the posterior pair of the Suprabranchial arteries is the first branch from the Dorsal aorta. This is a pair of small vessels (Pl. VI, Ph. Ar.) running backwards and giving branches to the lateral and postero-dorsal walls of the pharynx. About 1·3 cms. behind the last pair of the efferent branchials is given off the second pair of arteries, *viz.*, the Subclavians (Sbcl. Ar.) from the Dorsal aorta. Each of these arteries runs directly outwards and backwards to the base of the Pectoral fins where it divides into three or four branches supplying the muscles of the fin. The artery arising behind the subclavians, *i.e.*, the Coeliaco mesenteric (Coel. mes. Ar.), is the largest of the series of vessels given off from the Dorsal aorta. It is a median branch and placed at a distance of 6 mm. from the Subclavian arteries, and supplies nearly all the important viscera except the kidneys. Behind the point of origin of the Coeliaco-mesenteric artery the Dorsal aorta is slightly compressed dorso-ventrally and runs backwards in a shallow groove lying on the under surface of the abdominal part of the vertebral column.

Now to describe the exact course of the Coeliaco-mesenteric artery. At the anterior region of the coelomic cavity the artery passes downwards in between the ventro-lateral processes of the anterior vertebrae and then runs along the side of the alimentary tract. The major part of the artery is hidden from view owing to its being enveloped by the Gas-bladder on its dorsal side and by the liver mass on its ventral side. Nearer the junction of the oesophagus with the stomach the coeliaco-mesenteric divides into two branches, *viz.*, the coeliac artery (Pl. VII, Coel. Ar.) and the mesenteric artery (Mesen. Ar.),—the former mainly supplies the stomach (Gast. Ar_1), besides giving

a small branch to the oesophagus (Oes. Ar.) and another to the right lobe of the liver (Hep. Ar. R.), whilst the course taken by the latter may be described as follows.

The mesenteric artery runs backwards along the intestine, and gives a large branch which divides into two twigs one of which is small and supplies the left lobe of the liver (Hep. Ar. L.) while the larger one, after giving a few branches to the stomach (Gast. Ar_2), is continued onward over the intestine which it supplies with fine branches. The mesenteric artery after running for a distance of nearly 3 cms. from the point of origin of the coeliac artery divides into three branches, *viz.*, the two lateral ones supplying the gonads (Gen. Ar.) and the median one, the Gas-bladder. The Gas-bladder artery (Ar. bl. Ar.) runs backwards along the ductus pneumaticus (Pn.D.) and about 5·5 mm. from the rete mirabile (Ret. mir.) it divides into two branches (Bl. Ar. R. & Bl. Ar. L.) which run along the dorso-lateral walls of the posterior moiety of the Gas-bladder, each ending towards its posterior end into a bunch of capillaries distributed over its wall. Slightly behind the point of bifurcation of the Gas-bladder artery into two branches supplying the posterior moiety of the organ small arteries are given off from the latter, one on each side, which provide pure blood to the lateral body muscles. After giving off the coeliaco-mesenteric artery the aorta pursues its course backwards, as already stated, along the ventral surfaces of the vertebrae; on its course it gives off several small segmental arteries (Pl. VIII, a-k) which are more or less symmetrically arranged and supply the lateral body musculature. It should be marked here that the part of the Dorsal aorta situated in the anterior part of the coelomic cavity is easily seen from a ventral dissection, but its posterior part gets buried within the thickened region of the renal mass.

Towards the anterior region of this renal area two large vessels are given off, *viz.*, the right (Pelv. Ar. R.) and the left Pelvic (Pelv. Ar. L.) arteries situated at a distance of 4·8 cms. and 5·5 cms. respectively from the origin of the coeliaco-mesenteric artery. They pass through the kidney substance and finally supply the muscles of the Pelvic fins. The Pelvic arteries, while traversing the renal mass, give off a few small renal affluents as do the sciatics and the femoral of certain other vertebrates. The proper renal arteries are normally three in number two of these being on the right side

PLATE V

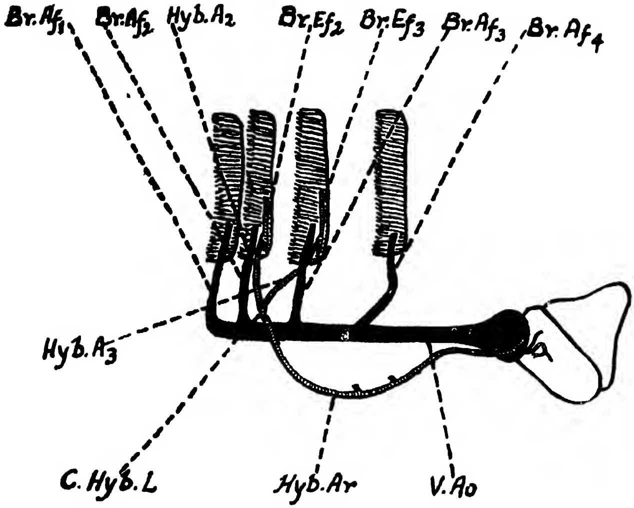


Fig 1

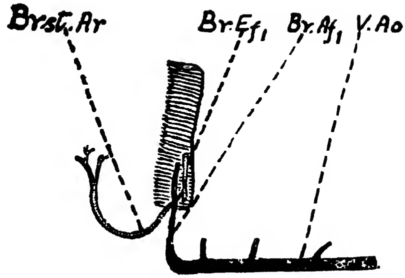


Fig. 2

(Ren. Ar_1 and Ren. Ar_2) and only one on the left side (Ren. Ar. L.). The two right renal arteries arise separately at a distance of 6 mm. and 1·1 cms. from the right Pelvic artery, while about 1·1 cms. from the left Pelvic artery the single left renal artery is given off. Posteriorly after emerging from the kidneys the Dorsal aorta is continued backwards as the caudal artery (Caud. Ar.) lying within the haemal arches. Corresponding to the anal fin vein there is an anal fin artery which arises from the caudal artery soon after it enters into the haemal arches.

SUMMARY.

The foregoing pages may be summarised as follows :—

1. As usual the heart consists of the auricle, the ventricle and the sinus venosus. The inner walls of the sinus are smooth, whilst those of the auricle possess a large number of trabeculae and those of the ventricle, trabeculae and numerous muscular ridges and folds. The conus arteriosus is abortive. There are a couple of membranous valves guarding the sinu-auricular aperture, eight peculiar semilunar valves at the auriculo-ventricular aperture, and also a pair of semi-lunar pocket valves in the conus. The heart is supplied with pure blood by the branches of the Hypo-branchial artery.

2. Normally the fish has four afferent branchial vessels and a corresponding number of efferent vessels. The afferent vessels open separately into the ventral aorta, whilst the efferents unite in pairs on either side to form the anterior and posterior supra-branchial arteries, which later unite with their fellows of the opposite sides giving rise to the Dorsal aorta. This condition of the efferent system is a case of specialisation. From the first efferent vessel are given off the external and internal Carotids,—the latter forming the Cerebral artery and the Circulus cephalicus.

3. There is a spiracular Pseudobranch and associated with this are the " Hyoidean " (given off from the first efferent) and the Ophthalmic arteries. There is a median Hypobranchial artery formed by the branches derived from the second and the third efferent branchials. It lies on the ventral surface of the ventral aorta and supplies the heart muscles and the gills. A small branch is also given off from the first efferent, supplying the branchio-stegals.

4. The afferent branchial vessel is situated ventral to the efferent vessel within the groove of the gill-arch and they give out fine twigs which capillarise over the gill-filaments.

The plan of the efferent system is intermediate between the African Cat-fish, Malopterurus and the common Carp, *Cyprinus Carpio*, and links with that of the European Loach, *Cobitis taenia*.

5. The principal arteries are given off from the dorsal aorta in the following order :—

(*a*) One pair of small arteries, going to the lateral and postero-dorsal walls of the pharynx.

(*b*) A pair of Subclavians.

(*c*) Single median coeliaco-mesenteric artery which divides into the coeliac branch supplying the stomach, oesophagus and the right lobe of the liver, and the mesenteric branch supplying the left lobe of the liver, stomach and intestines as well as the gonad and the gas-bladder.

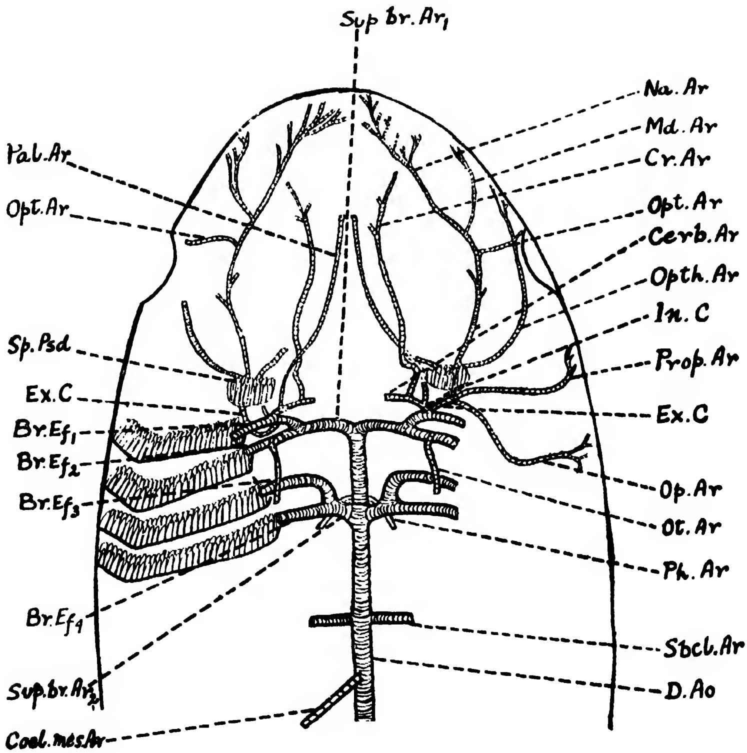
(*d*) Several pairs of small segmental arteries more or less symmetrically disposed.

(*e*) A pair of Pelvics.

(*f*) Two right Renal arteries and a single left Renal.

(*g*) Caudal artery, giving off the small anal fin artery.

PLATE VI



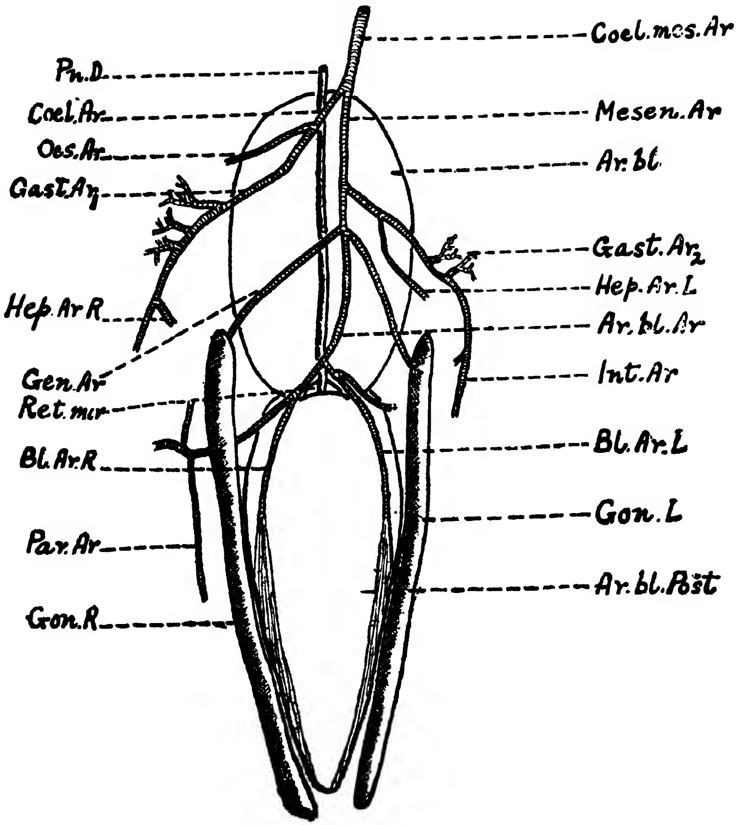
Sup br. Ar1
Na. Ar
Md. Ar
Cr. Ar
Pal. Ar
Opt. Ar
Opt. Ar
Cerb. Ar
Opth. Ar
In. C
Sp. Psd
Prop. Ar
Ex. C
Ex. C
Br. Ef1
Br. Ef2
Br. Ef3
Op. Ar
Ot. Ar
Ph. Ar
Br. Ef4
Sbcl. Ar
D. Ao
Sup. br. Ar2
Coel. mes Ar

LIST OF ABBREVIATIONS USED

Accr. V	...	Accessory semi-lunar pocket valve.
Ar. bl.	..	Air bladder.
Ar. bl. Ar.	...	Air bladder Artery.
Ar. bl. Post	...	Posterior moiety of the Air bladder.
Au.	...	Auricle.
Au. Vt. V.	...	Auriculo-ventricular, semi-lunar pocket valve.
Bl. Ar. L.	...	Left branch of the Air bladder Artery.
Bl. Ar. R.	...	Right branch of the Air bladder Artery.
Br. A	...	Branchial Arch.
Br. Af.	...	Afferent Branchial Vessel.
Br. Af_1	...	First Afferent Branchial Vessel.
Br. Af_2	...	Second Afferent Branchial Vessel.
Br. Af_3	...	Third Afferent Branchial Vessel.
Br. Af_4	...	Fourth Afferent Branchial Vessel.
Br. Af. fl.	...	Afferent twig to the Branchial filament.
Br. Ef.	...	Efferent Branchial Vessel.
Br. Ef_1	...	First Efferent Branchial Vessel.
Br. Ef_2	...	Second Efferent Branchial Vessel.
Br. Ef_3	...	Third Efferent Branchial Vessel.
Br. Ef_4	...	Fourth Efferent Branchial Vessel.
Br. Ef. fl.	...	Efferent twig from the Branchial filament.
Brst. Ar.	...	Branchiostegal Artery.
Bulb. Art.	...	Bulbus Aortae.
Caud. Ar.	...	Caudal Artery.
Cerb. Ar.	...	Cerebral Artery.
C. Hyb. L.	...	Left factor of the Hypo-branchial Artery.
C. Hyb. R.	...	Right factor of the Hypo-branchial Artery.
Coel. Ar.	...	Coeliac Artery.
Coel. mes. Ar.	...	Coeliaco-mesenteric Artery.
Con. Art.	...	Conus Arteriosus.
Con. V.	...	Valve in the Conus.
Cr. Ar.	...	Dorso-lateral Artery of the mouth cavity.
D. Ao.	...	Dorsal Aorta.
Ex. C.	...	External Carotid Artery.
Gast. Ar_1 / Gast. Ar_2	...	Gastric Arteries.

Gen. Ar.	...	Genital Artery.
Gon. L.	...	Left Gonad.
Gon. R.	...	Right Gonad.
Hep. Ar. L.	...	Left Hepatic Artery.
Hep. Ar. R.	...	Right Hepatic Artery.
Hyb. A_2	...	Hypo-branchial derivative from the second Efferent.
Hyb. A_3	...	Hypo-branchial derivative from the third Efferent.
Hyb. Ar.	...	Hypo-branchial Artery.
In. C.	...	Internal Carotid Artery.
Int. Ar.	...	Intestinal Artery.
Md. Ar.	...	Mandibular Artery.
Mesen. Ar.	...	Mesenteric Artery.
Mesonph	...	Mesonephros.
Na. Ar.	...	Nasal Artery.
Oes. Ar.	..	Oesophageal Artery.
Op. Ar.	...	Opercular Artery.
Opt. Ar.	...	Artery to the eye muscles.
Opth. Ar.	...	Ophthalmic Artery.
Ot. Ar.	...	Artery to the muscles of the Otic region.
Pal. Ar.	...	Artery to the Palate.
Par. Ar.	...	Parietal Artery.
Pelv. Ar. L.	...	Left Pelvic Artery.
Pelv. Ar. R.	...	Right Pelvic Artery.
Ph. Ar.	...	Pharyngeal Artery.
Pn. D.	...	Pneumatic Duct.
Prop. Ar.	...	Preopercular Artery.
Ren. Ar_1	...	First Right Renal Artery.
Ren. Ar_2	...	Second Right Renal Artery.
Ren. Ar. L.	...	Single Left Renal Artery.
Ret. mir.	...	Rete mirabile.
Sa	...	Sinu-auricular aperture.
S. Au. V.	...	Sinu-auricular Valve.
Sbcl. Ar.	...	Subclavian Artery.
Sp. Psd.	...	Spiracular pseudobranch.
Sup. br. Ar_1	...	Anterior Suprabranchial Artery.
Sup. br. Ar_2	...	Posterior Suprabranchial Artery.
S. V.	...	Sinus Venosus.

PLATE VII

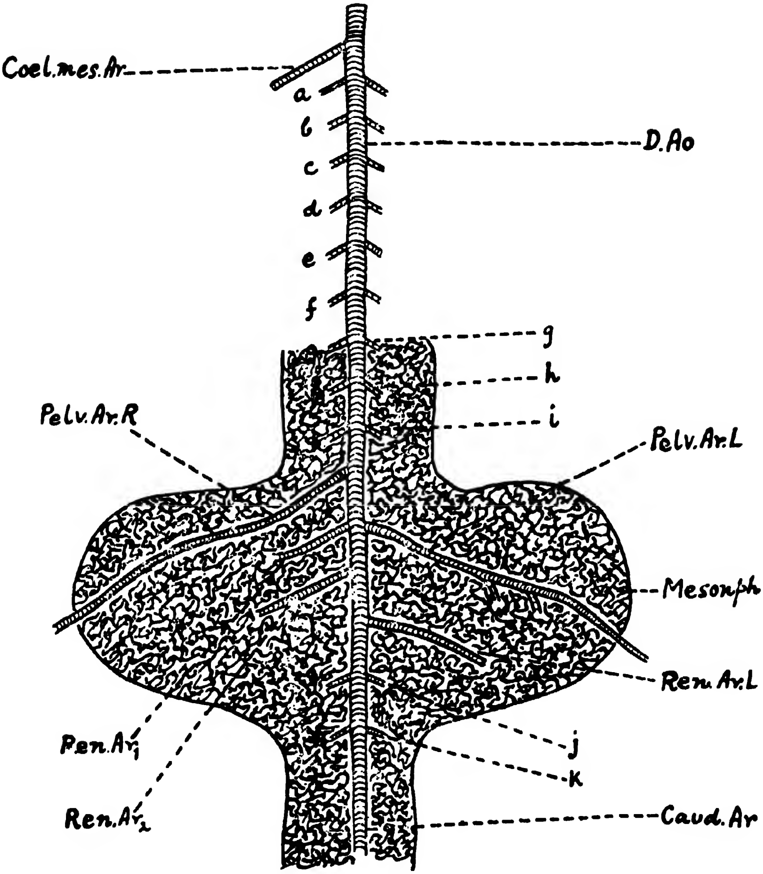

Coel.mes.Ar
Pn.D
Coel.Ar
Mesen.Ar
Oes.Ar
Ar.bl
Gast.Ar1
Gast.Ar2
Hep.Ar.L
Hep.Ar R
Ar.bl.Ar
Int.Ar
Gen.Ar
Ret.mir
Bl.Ar.L
Bl.Ar.R
Gon.L
Par.Ar
Ar.bl.Post
Gon.R

Va	...	Auriculo-ventricular aperture.
V. Ao.	...	Ventral Aorta.
Vent.	...	Ventricle.
a—k.	...	Intersegmental Arteries.
1—8	...	Auriculo-ventricular semilunar pocket valves.

EXPLANATION OF PLATES.

Pl. I (× 2)	..	Heart of 'Rohu' seen from various aspects.
Fig. 1	...	Lateral View.
Fig. 2	...	Ventral View.
Fig. 3	...	Dorsal View.
Pl. II (× 2) Fig. 1	...	Heart dissected from the ventral surface showing the apertures and the valves.
Fig. 2	...	Disposition of the sinu-auricular valves after opening the auricle from the dorsal side.
Fig. 3	...	Mode of arrangement of the auriculo-ventricular semilunar pocket valves.
Pl. III (× 2)	...	Afferent Branchial system of 'Rohu.'
Pl. IV (× 2)	...	Disposition of the Hypo-branchial Artery seen from a ventral dissection.
Pl. V (× 2), Fig. 1	...	Semi-diagrammatic arrangement and mode of distribution of the Hypo-branchial artery as seen from the lateral aspect.
Fig. 2	..	Branchiostegal artery dissected out to show its distribution.
Pl. VI (× 2)	...	The Efferent Branchial system and the Anterior arteries derived from the same as well as from the Dorsal Aorta seen from a ventral dissection.
Pl. VII (× 1)	...	Distribution of the Coeliaco-mesenteric Artery.

PLATE VIII



Coel.mes.Ar
a
b
c
d
e
f
D.Ao
g
h
i
Pelv.Ar.R
Pelv.Ar.L
Mesonph
Ren.Ar.L
Ren.Ar1
Ren.Ar2
j
k
Caud.Ar

Pl. VIII (× 2)	...	Course of Dorsal Aorta behind the Coeliacomesenteric artery, showing the mode of origin of the principal arteries from it.
Pl. IX (× 14)	...	Diagrammatic representation of the branchial circulation (modified after Agassiz and Vogt). Gill-arches and Gill-filaments cut across.

BIBLIOGRAPHY.

1. Agassiz, L. & Vogt, C. ... Anatomie des Salmones. Mem. de la Soc. des Sci. Naturelles de Neuchatel. Tome III, 1845.

2. Allis, E. P. (Jr.) ... The Pseudobranchial and Carotid Arteries in the Gnathostome Fishes. Zool. Jahrb., Jena, Abt. F. Anat. 27. 1908 (103-134).

3. Allis, E. P. (Jr.) ... The Pseudobranchial and Carotid Arteries in Esox, Salmo and Gadus, together with a description of the arteries in the adult Amia. Anat. Anz., Jena, 41. 1912 (113-142).

4. Bridge, T. W. ... "Fishes." Cambridge Natural History, 1910 (141-420).

5. Burne, R. H. ... Note on the Cerebral Arteries of the Angler Fish (*Lophius piscatorius*). Proc. Zool. Soc., London, 1925 (871-877).

6. Cuvier, B. Le M. & Valenciennes, M.A. ... Histoire Naturelle des Poissones, Vol. 16. Paris. 1842.

7. Cuvier, G. ... Le Regne Animal. 1836 (220).

8. Danforth, C. H. ... The Heart and Arteries of Polyodon. J. Morph. Phil., 23. 1912 (409-454).

9. Day, F. ... "Fishes." Fauna of British India, Vols. I & II.

10. Einstmann, J. W. ... Über das Venensystem der einheimischen Teleostier. Arch. Natg. Berlin., 79, Abt. A. 1913 (1-31).

11. Gegenbaur, C. ... Über den Conus arterioses der Fische.
Morph. Jahrbuck, Vol. XVII. 1891.

12. Goodrich, E. S. ... "Fishes."
A Treatise on Zoology. London. 1907.

13. Günther, A. C. I. G. ... An Introduction to the Study of Fishes. Edin. 1880.

14. Günther, A. ... Catalogue of the Fishes in the British Museum, Vol. VII. London. 1868.

15. Hamilton, F. ... Fishes of the Ganges.

16. Mozejko, B. ... Über die Injektion des vascular systen von Petromyzon fluviatilis.
Zs. Wiss. Mikrosk., Leipzig, 27, 1910 (248-250).

17. McKenzie, T. ... The Blood Vascular System, Ductless Glands and Uro-genital system of Amiurus catus.
Proc. Canad. Inst. 1884. Vol. II (418-432).

18. Owen, R. ... "Fishes."
Lectures on Comparative Anatomy, London, 1846.

19. Parker, T. J. ... On the Blood Vessels of *Mustelus antarcticus*.
Phil. Trans. Roy. Soc., London, 1886.

20. Ridewood, W. G. ... On the relations of the efferent branchial vessels to the circulus cephalicus in Teleostean Fishes.
Proc. Zool. Soc. 1889 (939-956).

21. Rosen, N. ... Studies on the Plectognaths. (1) The Blood Vascular System.
Ark. Zool. Stockholm. 7, No. 25, 1902 (24).

22. Sagemehl, M. ... On the Anatomy of the Cyprinoids.
Morph. Jahrb. XVII. 1891. (489).

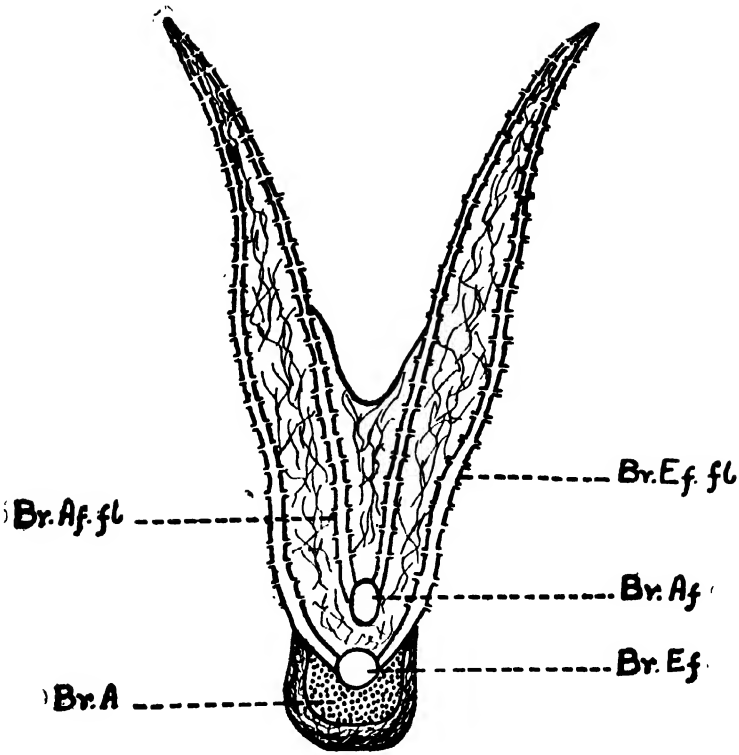
23. Silvester, C. F. ... The Blood Vascular system in the Tile Fish.
Bull. Bur. Fish. Woods Hole. Mass. 1904. Vol. 24.

24. Stannius, H. ... Handbuch der Anatomie der Wirbelthiere. Berlin. 1854.

25. Stewartzoff, A. N. ... Die Entwicklung der Kiemen und Kiemen bogen gefasse der Fische. Zs. Wiss. Zool. Leipzig. 121, 1923 (494-556).

26. Winterstein, Hans ... Handbuch der Vergleichenden Physiologie. Jena. 1921.

PLATE IX



On an *Octomitus* n. sp. found in the intestinal contents of *Hylobates hoolock* (with one plate).

By

Dr. G. C. Chatterji, M.B., Lecturer of Protozoology, C.U.

and

Mr. Khagendra Nath Das, M.Sc. (Gold Medalist)

and

Dr. A. N. Mitra, B.Sc., M.B.

As there is a certain amount of difference of opinion among observers regarding Octomitus, it is best to refer to it, before the description of the species observed by us is given. To Dujardin is attributed the first finding in 1841 of three species of eight-flagellate organisms, one parasitic in frogs and two free-living ones. Though he observed only six flagella in these organisms, four free and two attached, yet subsequent observers made out by examining these specimens, that Dujardin was really dealing with eight-flagellate organisms and not six-flagellate ones. Dujardin named the genus Hexamitus. Prowazek (1904) described Dujardin's parasite in frogs as an Octomitus, as also Dobell (1908) on account of these showing eight flagella. According to the rule employed by Parisi, flagellates having six free flagella and two attached ones, should be named Hexamitus and not Octomitus. So there is a consensus of opinion among most of the observers to call these as Hexamitus.

Octomitus found in man by Chalmers and Pekkola (1906) having six free and two attached flagella, is also a Hexamitus according to the above rule, though Wenyon puts down his opinion, after examination of the slide specimen, that this is nothing but a Trichomonas.

Now, the organism found by us is, according to the rules laid down by Parisi, a true Octomitus and not a Hexamitus inasmuch as it possesses eight free flagella and as it is, to all appearances, an organism which has not been observed before, it is a new species and so worth recording.

The material from which it is got is from the intestinal contents of a Hoolock (*Hylobates hoolock*) dying in the Zoological Gardens, Alipore, the dead body of which was sent to the University College Laboratory for examination. It is worth recording that along with these, were found in the intestinal contents of this particular animal, two apparently new species of Lophomonas and a Trichomonas having four flagella. We tried to cultivate these organisms. There was an abundant growth, but the culture on examination showed Trichomonas, the Lophomonas not growing and Octomitus growing sparingly.

Examination showed that the organism has got a bilateral arrangement and so belongs to the class Diplozoa. The body is oval and is 4μ to 6μ in length and 3μ to 4μ in breadth. It shows a single nucleus, situated in front in the median region. It is of massive character. There are seen two pairs of basal granules from which are seen originating two bunches of flagella, each bunch containing four free flagella. They are nearly all of equal size, in many specimens directed backwards, and in some specimens directed forwards in a single bunch. In the cytoplasm there are seen some vacuoles. No other structure was noticeable in the organism. No divisional forms were found.

Reference :

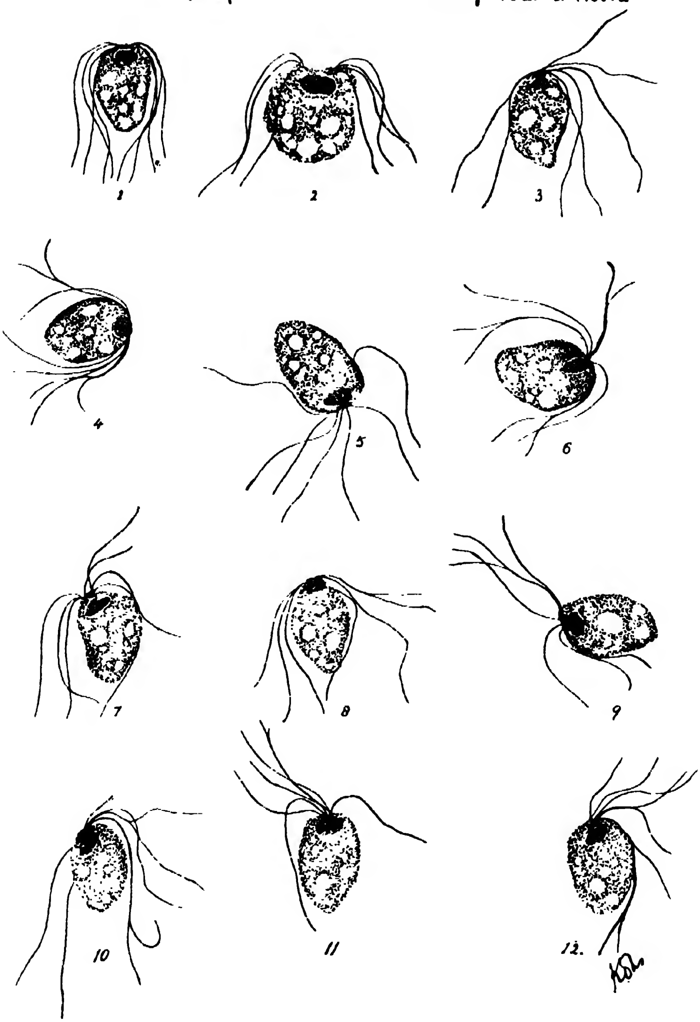
Besides the references about Octomitus given above, the only paper to which it is necessary to refer is the Octomitus described by one of us in 1922. This was found in the intestines of man. From this the organism of ours differs, as having a single nucleus and the flagella being arranged bilaterally and not in a circle around the nucleus.

Examination of the stained film :

The specimens were made after fixing wet film in Acetic Schaudin and stained by iron hæmatoxylin.

Figures were drawn under Camera Lucida and magnified 2,500 times.

Octomitus N. Sp.   Chatterjee. Das & Mitra



Description of the Plate:

Figs. 2, 7 and 11 — Show the basal granules arranged in two pairs.

Figs. 5 and 10 — Show the basal granules arranged in one lump.

The rest show no differentiation of the basal granules from the nucleus.

REFERENCES.

1. Chalmers, A. J., and Pekkola, W. (1916) — A New Human Intestinal Flagellate in the Anglo-Egyptian Sudan. 1916. Jour. Trop. Med. and Hyg., XIX, p. 142.
2. Chatterjee, G. C. (1922). — A New Flagellate Parasite in the Human Intestinal Canal. 1922. Ind. Jour. of Med. Research, X, No. 2, pp. 523-?5.
3. Dobell, C. (1908 *c*) — On the Intestinal Protozoan Parasites of Frogs and Toads. 1908 (*c*). Proc. Cambridge Philos. Soc., XIV, p. 428.
4. Dujardin, F. (1841) — Histoire naturelle des Zoophytes Infusoires comprenant la physiologique et la classification de ces anemaux. Paris.
5. Nöller, W., and Buttgereil, F. (1923) — Über ein neues parasitisches Protozoon der Haustaube (Octomitus columboe, n. sp.) Centralbl. Bakt. Ref. LXXV, p. 239.
6. Parisi, B. (1910) — Su alcuni flagellati endoparassiti. Arch. Protist. XIX, p. 232.
7. Prowazek, S. (1904 *a*) — Unter suchungen über einige parasitische Flagellaten. Arb. K. Gesundheitsamte, XX, p. 440.
8. Schimdt, W. (1920) — Untersuchungen über Octomitus intestinalis truttoe. Arch. Protist., XL, p. 253.
9. Wenyon, C. M. (1926). — Protozoology, London.

THE ALKALI SOILS OF INDIA.

BY

N. N. GANGULEE, D.Sc.,

KHAIRA PROFESSOR OF AGRICULTURE.

Introduction.

The object of this paper is to present before the International Conference of Pedalogy a brief outline of the problems of the 'alkali' soils of India. References have been made to the investigations carried out by the Government of India and the importance of further research is emphasized. In view of the fact that alkali phenomena are of common occurrence in the arid or semi-arid regions of the world and that the reclamation of cultivable land impregnated with saline deposits is of paramount importance for the food supply of the world's increasing population, the problems relating to soil alkali can justly claim the attention of this Conference.

Prevalence in India.

Long before the attention of the scientific investigators was directed to the appearance of incrustation of salt deposits on the soil surface of the arid and semi-arid soils of India, the Indian cultivator did not fail to observe the phenomenon, and regarded it as an indication of approaching sterility of his land. As he realised that they originated in the subsoil tract, he took care not to plough his land deep enough ; and, in order to control evaporation as far as possible, he practised mulching. It is common belief among the farmers of the regions that organic matters in excess must be added to change the nature of the soils containing soluble salts.

The term 'Alkali' generally refers to the saline efflorescences deposited on the surface of the ground, but the soils where they occur should be classified into two types in reference to the nature of the salts, one signifying the presence of soluble salts not harmful to plants and the other those that are toxic to plant growth.

Various vernacular names are given to alkali soils of India. The salts which accumulate on the surface are known as '*Reh.*' In the Punjab, the North-West Provinces, the Deccan and Sindh, saline

efflorescence is known as '*Kalar*' and the cultivator classifies it according to certain obvious characteristics of the deposits. For example, *Rapri Kalar* (refers to chlorides), *Bhanothi Kalar* (probably refers to sulphates) and *Dudh Kalar* (refers to carbonates). In other parts of India, the alkali lands are called *Usar* (meaning sterile, unfertile), and the cultivator distinguishes the predominant types of salts as *Sajji* (sodium carbonate), *Khari* (sodium sulphate) and *Namak* (sodium chloride).

Industrial Uses.

Wherever alkali efflorescence is obtained in large quantities, the salts are put to many uses by the people. They are employed in the processes of dissolving crude lac and for extracting the dye from Safflower. Watt (1) refers to the practice of sprinkling white *Reh* into boiling cane juice to neutralise the organic acids in the manufacture of sugar. *Khari* (sodium sulphate) is used for medicinal purposes and also as a preservative for hides. The crude method of extraction of alkalis from the efflorescent deposits on *usar* lands is practised by a class of people known as *Lunia*, and the local industries such as the manufacture of glass bangles and soap are thus supplied with sodium salts; but since the advent of English alkalis the market for these crude sodium salts has considerably shrunk and is mostly confined to remote villages. The indigenous methods may be briefly described as follows :—

The saline efflorescent is scraped off the ground and is dissolved in water. By an ingenious process of filtration through earth, the filtrate is freed from humus and other matters; it is then allowed to evaporate and soluble salts are collected. During the War, owing to the shortage of alkali imports, attempts were made to revive and improve the indigenous method of extraction.

Estimated Area.

In the absence of any reliable statistics or survey, it is not possible to quote figures showing the extent of alkali soils in India; but its extensive prevalence throughout the semi-arid and arid regions cannot but arrest the attention of a soil investigator. Voelcker, in his report (2) estimates that "in the North-West Provinces alone there are between four and five thousand square miles of *Usar* land" and Hills' (3) estimate for the whole of India comes to over two million

acres in 1903. It may be safely assumed that of the vast area[1] recognised in the official statistics as "culturable waste other than fallow," a very large proportion of land is impregnated with alkali salts. The area recorded in 1920-21 as "culturable waste other than fallow" in some of the provinces where saline efflorescences are known to occur is given below :

Provinces.	Cultivated area (in acres).	Culturable waste other than fallow (in acres).	Culturable waste in Percentage.
Sindh ...	3,564,365	6,072,339	170
North-West Provinces ...	1,691,792	2,864,454	169
Central Provinces ...	16,523,904	14,351,902	86·8
Punjab ...	21,788,478	16,264,567	74·6
Ajmer-Merwara ...	315,105	207,487	65·6
Oudh (United Provinces) ...	9,033,621	2,860,776	31·6

Rainfall.

The general rain-map of India shows the area covering the whole of the Punjab and Rajputana and Sindh to be arid throughout. The rainfall of the plains of the Punjab decreases eastward. With the rainfall above 30″, the United Provinces and the Deccan may be characterised as the semi-arid regions on account of excessive evaporation which leaves land quite dry. The normal rainfall of the above regions is given below :—

	(Inches.)
North-West Frontier Province ...	4·9
Sindh	4·8
Punjab	15·5
Rajputana	17·4
Deccan	31·5
Central India	33·3
United Provinces	36·2

The Investigations by the Government.

One of the discouraging features of extending cultivation in alkali regions is that, although it essentially depends on irrigation, its

[1] In 1920-21, the area reported under this head was 114,848,090 acres in British India.

introduction appears to disturb the equilibrium of the underground water level, causing "the rise of the alkali." The Irrigation Department of the North-West Provinces observed this distressing phenomenon, and in response to the petition of the planters, a Commission, known as the *Reh* Commission, was appointed in 1876 "to discover the cause of deterioration of some of the lands that had previously been fertile." Although the alkali phenomena caused great anxiety to the cultivator of the affected tracts and to the Irrigation authorities, the matter unfortunately was investigated in a more or less cursory manner, and no investigation of any importance to agriculture was carried out by the various experiment stations in India except the ones mentioned here. Leather's analytical work (4) of a number of soil samples from the canal areas of the United Provinces furnishes certain data as regards the correlation between the soil types and the alkali formation. Barnes, in the Punjab, made an investigation of some biochemical factors involved in the reclamation of alkali soils and Henderson (5) experimented on the possibility of adapting the Egyptian methods of reclamation in Sindh. Mann's investigations, (6) both in the Deccan and in Sindh, throw much light, not only on the problems prevailing in those regions, but also on the probable factors which may determine the alkali phenomenon.

Until recently the alkali investigations had been largely dominated by the school of Hilgard. With the development of physical chemistry, it is now known that both adsorptive and absorptive forces are at work in soils in bringing about such complex phenomena as the salt accumulation. The mechanism of base exchange requires further elucidation and in this direction systematic research should be directed by the Indian Agricultural Department.

Theories regarding its Origin.

Many theories are advanced as regards the possible explanations of the alkali phenomena on the arid and semi-arid regions of India Medlicott (7), Superintendent of Geological Survey of India, gave his opinion before the 'Reh' Commission as follows :

"Every one seems content to accept reh as an ultimate fact that there it is—as much an original ingredient of the ground as the clay and sand. Now, I consider it demonstrated that this view is false, and that an origin is assignable for reh which introduces the gravest apprehensions as to the possible results of canal irrigation. The

present chief store of reh is in the saline subsoil water—the upper water-bed found more or less all over Upper India. Beneath this, at depths of from sixty to one hundred feet, sweet water is found, containing no more salt than the canal water, which is very pure. Now, I consider it certain that these upper deposits were originally as free from those salts as is now the alluvium of the delta; and that the present state has been brought about in the course of generations slowly, but with increasing rapidity, owing to the total subversion of the natural climatic conditions, chiefly by the total destruction of forest vegetation. The explanation is simple: every soil contains some reh, and all percolation water from soils is also contaminated by those salts, which are the refuse products (the parts unassimilated by vegetation) in the very slow process of formation and consumption of soil. Unless removed it must accumulate; and the natural process of purification is a certain necessary amount of percolation and ground drainage, the pure rain water passing through the soil carrying off any injurious excess of these rejected salts. If the washing process is sufficiently free to insure a certain discharge of this percolation water by natural subsoil drainage, there will be a constant dilution and removal of the subsoil water; but if the percolation through the soil is no more than to restore what has been dissipated by capillary action and evaporation during the dry season, the reh will go on accumulating in the upper water table; and such has been the condition of Upper India for many a day."

Ball, on the other hand, maintains (8) that the saline deposits are formed by normal process of weathering, leaving high concentrations in the lower depths of the regions of limited rainfall. The processes of formation described by him are quoted below:

"Primarily the saline matters are derived from the decomposition of rocks, and, taking the case of Northern India, the rivers descending from the Himalayas carry down in solution proportions of salt which vary with the character of the strata traversed. The salts so carried in solution consist principally of calcium and magnesium carbonates and sodium sulphate and chloride. In addition, of course, the alluvium or silt which is brought down, consisting of finely comminuted minerals, includes materials which, on decomposition, are capable of supplying bases for the ultimate formation of the same salts under suitable conditions. In a region of intense evaporation,

and where there is not a free drainage outlet of water, these salts, by long continued concentration, accumulate in the soil or the subsoil waters and over and above this, rain water charged with carbonic acid falling on a porous soil, has the effect of decomposing its mineral constituents and of carrying down the salts so formed in solution either to the region of subsoil water, or else for only a few inches or feet below the surface. When the surface of the ground again becomes dry, this saline water rises by capillary attraction, and evaporates, and a salt efflorescence remains, which at length so permeates the superficial layer of soil that cultivation becomes impossible. With free underground drainage, which would admit of the rain passing through and washing the soil, this would not occur, especially where the surface was well-protected from evaporation by vegetation."

In view of certain geological evidence that during the Tertiary period an Eocene sea covered the Punjab plain with salts, it is thought that the saline efflorescence so common in the Punjab and Indo-Gangetic basin have their origin in such marine deposits.

The Commission (9) arrived at the general conclusion that if the water level were raised, the alkali would soon make its appearance on the surface as the result of evaporation. While canal irrigation may be a predominant factor in altering the conditions of the underground water, it was clear from the evidences that canal water did not itself bring alkaline salts to the land.

Leather, working in the United Provinces, comes (4) to the conclusion that " the alkali in the soil is a product of the soil itself and is not deposited by the canal water." Howard (10), who made observations in Beluchistan, Sindh and in the canal-irrigated tracts of North-Western India, suggests that defective soil aëration seems to be responsible for the production of alkali soils. He finds that the deposits are " accompanied by the formation of the bluish-green iron-compounds so characteristic of ill-drained soils. Subsequent oxidation appears to result in the formation of white alkali." Mann's recent investigations in the Deccan show (6) that while the alkali salts are formed by the decomposition of the rocks from which the soil is composed, their appearance on the surface is always associated with the rise of saline subsoil water. But in Sindh it appears that the soil itself is impregnated with the salts and the subsoil water is not responsible for its accumulation on the surface.

Irrigation and 'Alkali' Land.

Whatever may be the underlying causes of saline efflorescence under the soil conditions in India, it is generally found to occur in those parts where the cultivation depends on irrigation and also where the natural surface drainage is bad or has in any way been interfered with by an obstruction, such as a railway embankment. "The Reh Commission brought out the fact that under the ancient systems of agriculture in India there was very little increase in the amount of soluble salts at the surface, but with the construction of large modern canals and the application of unnecessarily large quantities of irrigation water the increase in alkali was very rapid." (11)

Ball pointed out before the Commission that "irrigation by canal water, when not accompanied by deep drainage, has had the effect of increasing the amount of *reh* deposit and large tracts have been in consequence thrown out of cultivation."

Henderson (5) finds that the areas under perennial irrigation in Sindh are becoming unfertile due to alkali deposits and writes: "if the yield per acre be inquired into, it will be found, according to the evidence of a large number of cultivators, to be very greatly reduced since the opening of the canal." Mules, Collector of Karachi, writes: "A very striking instance of the movement of subsoil salt induced by the action of irrigation channels or floods is to be seen at Jacobabad. For one who remembers Jacobabad in the early seventies the change is very remarkable. Then the land in cantonments was sweet, the wells were sweet and the water good, the gardens were remarkable for their luxuriant growth and the vegetables raised therein were every bit as good as the best English vegetables. Now the whole place is salt, the wells are many of them useless, and the gardens have died away."

Mann's investigations in the salt lands of the Nira Valley show that after yielding, as was expected, excellent harvests for a period of five years, about five thousand acres were rendered more or less completely barren, and the indications are that the affected area is increasing. Previous to the introduction of canal irrigation, the presence of salt incrustation was unknown, in this tract. Irrigation is one of the most important factors in Indian agriculture and, therefore, with extension of modern perennial irrigation works in India, the whole problem of alkali soils must receive due attention from the agricultural investigators of India.

Some Characteristics of the 'Alkali' Soils of India.

There is no fundamental difference in the nature of saline efflorescence as found in India and elsewhere in the arid and semi-arid regions. Only the ingredients and their relative proportions are found to vary in accordance with various factors such as geological formations, movement of underground water, the rate of evaporation and climatic conditions. Of all the elements composing 'alkali' efflorescence, sodium, occurring as chloride, sulphate, and carbonate is the most common in India. The soda felspar is a common ingredient of the Himalayan rocks and its decomposition in soils may possibly account for the predominance of sodium salts both in the soils of the Indo-Gangetic plain and in the subsoil water of the region. In certain parts excessive quantities of sodium chloride and calcium carbonate are found in the soils exhibiting the tendency of forming what is known as 'black' alkali. As the soluble carbonates are particularly toxic to vegetation, the investigations as regards their formation in Indian soils should be undertaken.

The appearance of dark colour has always been associated with the presence of sodium carbonates in the soil, but Mann's investigation in Sindh shows that certain soil samples with characteristic dark colour had only a small quantity of sodium carbonate. The writer, working cn the effects of various salts on the solubility of organic soil contents, obtained evidences of the solvent action of other sodium salts, such as sodium sulphate and sodium chloride, on the organic substances present in the soil. The classification of 'alkali' as white and black is not satisfactory.

The capricious, irregular distribution of patches of 'alkali' soils is one of the features which strike the observer almost everywhere in the semi-arid and arid plains of India; in the midst of an excellent sugar crop, bald patches without any vegetation or with grasses generally found on alkali soils are often discovered. Sometimes in the immediate neighbourhood of a good crop-bearing area, barren tracts are visible. Such phenomena as these only emphasize the complexities involved in the systems of various salts associated with that of the system of moisture. In many cases carbonate is not all in evidence on the surface, but may be discovered below it.

Almost universally 'hardpan' is found in the regions of alkali soils, and it should not be confused with superficial calcareous pans

resting on friable soils of lateritic origin. The presence of such an intervening layer causes a great deal of obstruction to any improvement of the land. With the hard crust formed probably in combination with the colloidal substances and calcium carbonate and its limited permeability to water, the task of dealing with hardpans becomes extremely difficult. In some parts a layer of soil, thick enough for cultivation, accumulates and the local agriculturists consider the crust beneath as a protection against the loss of the rain water.

A peculiar concretionary form of limestone known as *Kankar* or *Ghutin* occurs chiefly in the subsoil area of many parts of the Indo-Gangetic plains and of the Deccan trap. These 'nodules' contain a high percentage of carbonate of lime ranging from 25 to as high as 70 per cent.

Physical Conditions of the 'Alkali' Soil.

While the behaviour of alkali salts depends on the nature of soils, it is well known that the physical conditions of the soil undergo certain radical changes by the presence of large quantities of soluble salts. To what extent such changes are brought about under Indian conditions, and how they influence the system on soil moisture we have no experimental data, but it is hoped that, with the expansion of the activities of our research stations these problems will receive due attention. The results of Mann's investigation in Sindh, however, indicate some interesting correlations between the soluble salt contents and their effect on the capillary power of soils. The attention may be drawn to the following table :—

TABLE I.

	Soil No. 4 Subsoil 6"-12"	Soil No. 13 Surface 0"-6"	Soil No. 14 Subsoil 6"-12"	Soil No. 19 Surface 0"-6"	Soil No. 20 Subsoil 6"-12"
I. Mechanical Analyses.					
Materials increasing stiffness.	40·26	56·18	50·68	24·47	33·06
Materials making soil lighter.	59·74	43·82	49·32	75·53	66·94
II. Total soluble salts,	1·86	0·32	0·44	0·36	1·84
III. Capillary rise in centimeters at the end of 15 days.	33·5	60·5	62·9	43·8	34·1

It appears that the increased clay contents of the soils do not always influence the capillary rise of water. Here the soil No. 19 has less amount of clay than in the case of No. 4 or No. 20, but its capillary power is more than both the samples. From a number of Mann's analyses, the writer has plotted two curves (fig. 1), one presenting surface and the other subsurface soils. It is clear that the soluble salts in these soils have exerted a depressing action on their capillary power.

Further investigations on various types as soils need to be made before any definite conclusion as to the possible action of soluble salts on soil aggregates may be arrived at. In Sindh soils according to Mann, the appearance of soluble salts is not due to the capillary rise of water from the subsoil water table, but is due to the downward and upward movement of the *superficial* water used for irrigation.

Leather's observation noted by Watt (1a) is interesting from the point of view of our present knowledge of the Soil Complex. While filtering soils in order to obtain the soluble salts present therein he observed that "certain soils could practically not be filtered. A little muddy water percolated through at first, but very soon the surface of the filter-cloth became coated with a perfectly impenetrable layer, and further filtration was then impossible. The soils that manifested this peculiarity were those most highly charged with carbonate of soda."

Biological Factors in 'Alkali' Soils.

There is increasing evidence that the micro-organic soil population exerts a considerable influence on the fertility of soil, and therefore any conditions which retard or inhibit their activities may decrease the productiveness of the land. The toxicity of alkali salts is known to affect profoundly the biological conditions of the soil. Taylor (12) adduced some evidences to show that the sterility of certain Bengal soils obtained from lands frequently inundated by brackish water was not due entirely to the physico-chemical effects of saline substances, but to the decreased bacterial activity. The extent to which the salts limit bacterial activity is however still unknown.

The biochemical factors in alkali soils received some attention from Barnes (13) in the Punjab. He found that micro-organisms associated with soil-fertility lived in a more or less

dormant state, but became active as soon as favourable conditions prevailed. Only in case of their being in alkali-rich soils for a long duration their vitality was impaired. He measured bacterial activity by the determination of such processes as the production of carbon dioxide, the rate of nitrogen-fixation, and the rate of nitrification of ammonia. The results reproduced in Table II throw some light on the biochemical activity of the soils he dealt with.

TABLE II.

Showing the milligrams of CO_2 given out from 200 grams of soil (Barnes).

Duration of the Experiment.	Alkali soil from Narwala (Punjab).	Alkali soil washed with water.	Sterile patch of recent origin.	Normal soil.
After 1 day ...	2·60	36·8	8·5	31·2
,, 2 days ...	1·80	69·6	22·0	37·9
,, 3 ,, ...	2·40	71·6	14·9	30·3
,, 4 ,, ...	1·30	80·0	27·2	67·3
,, 5 ,, ...	2·20	95·4	32·1	74·5
,, 6 ,, ...	1·10	98·0	57·2	94·3
,, 7 ,, ...	1·20	88·8	48·0	74·8
,, 8 ,, ...	1·40	81·4	15·1	42·8
,, 9 ,, ..	1·60	59·8	42·9	62·7
,, 10 ,, ...	1·40	40·8	40 0	45·1
,, 11 ,, ...	0·55	31·6	32·1	30·2
,, 12 ,, ...	1·20	27·2	25·9	25·9
,, 13 ,, ...	1·50	22·0	29·0	22·8
,, 14 ,, ...	1·40	18·2	28·8	18·2
,, 15 ,, ...	1·60	18·2	27·7	16·0
Total ...	23·25	839·4	481·4	681·0

Experimenting with Nitrogen-fixation by Azotobacter Chroococcum, the following interesting results are obtained:

		Mg. of nitrogen fixed by one gram of soil with one gram of mannite.
Soil from alkali land of Narwala	...	1·23
Sterile patch of recent origin	...	7·80
Normal soil	...	7·07

Two possibilities suggest themselves from these observations. It appears that the biochemical methods may be used to measure the toxicity of the alkali salts as affecting the fertility of the land and that in such soils as are used in the experiments, the limiting factor is the solvent-water.

It is needless to refer to researches made in this direction by several western scientists; but, in India the line of work initiated by Barnes should be continued. Lipman and Fowler (14) found that the rate of nitrification was greatly increased by leaching the soils treated with 500 parts per million of sodium carbonate, 1000 parts per million of sodium chloride and 2500 parts per million of sodium sulphate and mixed salts. As regards the effect of different degrees of 'alkali' concentrations on bacterial activities, the results obtained by various investigators show large discrepancies. Lipman (15) for example, found that 250 parts per million of sodium carbonate inhibited the ammonifying and nitrifying bacteria, but Kelley (16) put the maximum limit to 4000 parts per million. Unless standard methods are adopted, the comparative value of such biological investigators is considerably impaired.

The explanation of the limits of toxicity in different organisms probably lies in the antagonistic action between the anions of salts formed by the dissociation. Lipman (17) noticed such actions of sodium salts and also observed antagonism "between toxic and stimulating salts as well as between two toxic salts."

Brown and Johnson's investigation (18) on "effects of certain alkali salts on nitrification" confirms Russell's remark on the general question of toxicity. "Between nutritive effects and toxicity," writes Russell (19), "the margin appears to be narrow, and almost all the elements essential to plant nutrition are capable of producing toxic effects under other conditions." Brown and Johnson (18) found, to quote an instance, that "Combinations of various salts in non-toxic individual amounts in the presence of calcium carbonate became toxic to ammonification."

Reclamation Work in India.

Soon after the publication of the report of the 'Reh' Commission the Irrigation and Agricultural Departments of the North-West Provinces and Oudh undertook some experiments on reclamation of alkali land. Their early attempts were chiefly confined to the introduction of alkali-resistant vegetation, especially grasses which might be suitable for grazing purposes. The method of manuring heavily was also tried, but no conclusive results were obtained from those unorganised experiments. Leather's experiments on salt land in canal areas of the United Provinces met with a partial success, but the operations being too costly, they could not be extended.

The reclamation work of any practical importance was carried out by Barnes in Narwala (Punjab) and by Henderson in Daulatpur (Sindh). The former experiment aimed at removing the excessive salt by drainage, and the methods employed were proved to be successful. They are now to be tried on an extensive scale in the lower Bari Doab region, and it is estimated that the capital value of the reclaimed land will be increased by over six million sterling.

In Daulatpur, Henderson attempted to find if the Egyptian method as employed by the Aboukir Company on the site of the lake would be applicable to Sindh. Unless the supply of irrigation water is plentiful, no judgment can however be pronounced as to the effectiveness of the methods under the conditions prevailing in Sindh.

The Government has lately sanctioned an extensive irrigation project known as the Sukkur Barrage Scheme, and it is hoped that organised investigations of alkali soils will be pursued now.

Vegetation in Alkali Land.

An ecological survey of the arid regions of India would show a very wide limit of tolerance plants have for the alkali salts. The estimates thus obtained would be far more reliable than the attempts made in laboratories under totally different conditions to determine the toxic limits of an individual salt on plants. Further, such data throw no light on what may occur when alkali salts act in conjunction bringing into play the influences of the ions formed by the dissociation of the salts.

The study of plant associations in the regions affected by alkali is, therefore, of considerable importance both from the point of view pedalogy and botany.

The land rich in saline efflorescence nourishes no vegetation and is entirely barren. In some parts where the conditions are not so bad, soon after the monsoon certain species of grasses are seen to grow.

It may be interesting to mention here that the Indian cultivator of the regions, through his empirical knowledge, can often predict with certain accuracy the probable encroachment of "Reh," judging from the appearance of certain native vegetation known to be associated with alkali phenomena.

Some of the grasses found in the alkali regions of India are mentioned below :—

Common Name.	Scientific Name.
Salt bush	Atriplex nummularia.
Usar grass	Tetrapogon tetrastachys.
Kar usara	Sporobolus pallidus.
Kans grass	Saccharum spontaneum.
Candel grass	Andropogon laniger.
Ditto	Andropogon intermedius
...	Chloris tetrastachys.
Anjam	Pennisetum cenchroides.
Narri	Diplachne fusca.
Dub	Cynodon dactylon.

The plants belonging to the Natural Order Leguminosae preponderate in India, and they also invade the alkali tracts, especially those regions where among the salts sodium carbonate is not present in excess. In less affected areas, various forage crops belonging to the order have proved successful, and names of a few are given below :—

Common Name.	Scientific Name.
Shaftal	Trifolium resupinatum.
Serji	Meliolotus indica.
Wal	Dolichos lablab.
Guar	Cyamopsis psoralioides.
Chowli	Vigna catiang.

The introduction of Egyptian clover (Berseem, Trifolium alexandrinum) has been very successful in Sindh and also in some parts of the Punjab.

The Alkali Problem.

Surveying the wide field of research which has engaged the attention of numerous investigators one finds that a good deal could have been achieved, if certain accepted standard methods were adopted. Most of the early investigators dealt with soils as being composed of inert substances, and were under the impression that the conditions influencing their actions were static. Today we realise that we are dealing with numerous complex factors under dynamic conditions. Our knowledge of these factors, biochemical and physical, has also made a considerable advance during the last decade, and, therefore, it is opportune that in the light of recent researches the problems of alkali soils should be investigated in a comprehensive manner. While the phenomena caused by the interaction of acids and bases in soils are being carefully studied in the well-known research centres of Europe and America, the study of the alkali phenomena, which may give a suggestive clue to the nature of complex changes in soil processes, should receive a fresh stimulus.

Conclusion.

The writer wishes to emphasize that, both from the economic and scientific points of view, the problems of alkali soils should claim serious consideration from this Conference. While millions of acres are lying waste in the arid and semi-arid regions of Asia and America, Europe is not entirely free from alkali. It is reported from the lower valley of the Po and also of the "Szik" lands in Hungary; but in Asia and America the destiny of millions rests on the reclamation of alkali lands, and the writer is convinced that the perennial scourge of distress from which India suffers may be greatly relieved if the vast plains of that country, now lying barren, could be transformed into arable lands.

In conclusion, the writer wishes to submit to this Conference one or two proposals, and hopes they will receive attention from such a representative assembly of the pedalogists of the world.

The suggestions are :—

(1) That a special Committee be formed to consider the problem of alkali investigation.

(2) That standard methods, whenever feasible, be adopted in conducting alkali researches.

(3) That analytical results be reported in terms as may be determined by the Special Committee instead of the conventional manner now in vogue.

References.

(1) Watt.—The Commercial Products of India, p. 55. John Murray, 1908.

(2) Voelcker.—Improvement of Indian Agriculture, Official Report, 1895.

(3) Hill.—Analysis of Reh. Proc. Chem. Soc. London, 19 (1900) No. 262, pp. 58-61.

(4) Leather.—Investigation of Usar land in the United Provinces, Government Publication, 1914.

(5) Henderson.—Alkali Experiments and Reclamation in Sindh, Department of Agriculture, Bombay, Bul. 64, 1914.

(6) Mann and Tamhane.—Salt Lands of the Nira Valley, Department of Agriculture, Bull. 39, 1915.

(7) Medlicott.—Note before the Reh Commission 1876. Quoted by Hilgard, Cal. Agri. Expt. Sta. Rept. 1890.

(8) Ball.—Economic Geology, Vol. III, p. 696. See also his report before the Reh Commission of 1876.

(9) Indian Government.—Reh Commission, 1876.

(10) Howard.—Agricultural Development of Baluchistan, Pusa Bulletin, No. 119 (121).

(11) Harris.—Soil Alkali. John Wiley Pais, 1920.

(12) Taylor.—Effect of Salts on Soil. Department of Agriculture, Bengal, Quart. Jour. 2 (1909), pp. 281-287.

(13) Barnes.—Alkali Soils : Some Biochemical Factors in their Reclamation. Agr. Jour. India, 12 (1917), pp. 368-389.

(14) Lipman and Fowler.—Preliminary Experiments of some effects of Leaching on Soil Flora. Soil Sc. 1, (1916), pp. 291-297.

(15) Lipman.—Toxic Effects of Alkali Salts in Soils on Soil Bacteria. Centbl. F. Bakt. Abt. 2, Bd. 33 (1912), pp. 305-313.

(16) Kelley.—Nitrification in Semi-arid Soils. Jour. Agri. Res. 7. (1916), pp. 471-437.

(17) Lipman.—Antagonism between anions as affecting ammonification in Soils. Centbl. f. Bakt. Abt. 2, Bd. 36, (1913), pp. 382-394.

(18) Brown and Johnson.—Effects of Certain Alkali Salts on Nitrification. Iowa Sta. Res. Bull. 44 (1918).

(19) Russell.—Soil Conditions and Plant Growth. Longmans Green & Co.

Summary.

In this paper a brief outline of the problems of the alkali soils of India is given and references have been made to the enquiries and investigations carried out by the Government of India. Theories regarding its origin in India are discussed, and certain characteristics of the saline efflorescence are indicated. Considering the importance of reclamation work in India, it is urged that organised research should seriously be undertaken by the Indian Experimental Stations. As alkali soils occur in the arid and semi-arid regions of North America, Canada, United States, Egypt and India, the author suggests that the International Conference of Pedalogists appoint a special Committee to consider the 'alkali phenomena' both from scientific and practical points of view and he is of the opinion that standard methods, whenever feasible, should be adopted in the investigations so that results may be interpreted to the best advantage of all concerned.

ON THREE DEEP-SEA DEPOSITS FROM THE BAY OF BENGAL

BY

N. N. CHATTERJEA, M.Sc.,
Lecturer on Geology, Calcutta University

(Read in the Geology Section of the Indian Science Congress held at Lahore, January, 1927.)

The three samples of deep-sea deposits which are the subject-matter of the present note were handed over to me by Prof. Das-Gupta who obtained them from Major Sewell, the Director of the Zoological Survey of India and I wish to thank them for the opportunities afforded to me to carry on the work and for the help and suggestions obtained from them.

The samples were collected from the deep soundings in the Bay of Bengal by the party on board the R. I. M. S. "Investigator." One of the samples as far as it could be deciphered from an ill-preserved label was from a great depth beyond 2,200 fathoms and only a few cases of ooze from such a depth of the Bay of Bengal have hitherto been recorded.

In their report on the deep-sea deposits Murray and Renard described a method of determining the percentage of lime.[1] This method does not appear to give any accurate results as the apparatus used was a very simple one without any precautionary measures to guard against the possible sources of error. Accordingly while working these deposits special care was taken for the estimation of CO_2 and instead of following Murray and Renard in this respect the method fully described in Clowes and Coleman's Practical Chemistry for the determination of the slightest amount of CO_2 [2] in any specimen was adopted with slight modification, as instead of using two-holed rubber cork for the conical flask, the writer made use of one with three holes. The first hole was for inserting the thistle funnel containing dilute hydrochloric acid (1 : 3), the second was connected with a train of absorption

[1] Report of Deep-Sea Deposits, p. 15, 1891.

[2] Quantitative Chemical Analysis, p. 104, 1900.

tubes and through the third was inserted one bent glass-tube right down to the bottom of the flask and dipping into the liquid. The outer end of the bent tube was connected with the U tube containing soda-lime to eliminate CO_2 from the air which would come out as bubbles through the liquid in the flask when the aspirator would work during the last stages of the experiment.

The physical properties of the samples were at first noted and the samples were thoroughly stirred with water in a flask with a long neck and the tiny parts of the deep-sea organisms were almost completely eliminated by means of fractional decantation as described by Murray and Renard.[1] These organisms were afterwards put under the microscope for their identification. Before the estimation of CO_2 the specimens were thoroughly dried in an air-oven between 105°C and 110°C.

Specimen I.

The data regarding the locality and the depth of the sample are, unfortunately, not very complete and as far as it could be deciphered from the ill-preserved label attached to it, it is clear that the depth was between 2,200 and 2,300 fathoms and the latitude and the longitude of the place are 6°N and 85° 07′ 10″ (?) E., respectively. These would place the spot between Ceylon and the Nicobar Islands but nearer Ceylon and on the border between the Bay of Bengal and the Indian Ocean if not in the open ocean itself.

The ooze in the dried condition is greyish white in colour, fine-grained, homogeneous and uniform in appearance. It is very light, and porous to some extent. Gritty particles are not perceptible when the substance is passed between the fingers. It is pulverulent though cohering slightly. Some of the foraminifera are visible to the naked eye as minute milk-white specks in the mass and they can be clearly seen with the help of a pocket lens. When the substance is powdered and acted on by dilute acids it effervesces briskly leaving behind some insoluble residue of a light brown colour. The substance is not brittle and the streak is greyish white. When wet it is plastic to some extent, the plasticity being probably due to the clayey matter present. When dried it has the property of adhering to the tongue. If a small fragment is placed in a little water it breaks down at once to fine powder and few black particles can be seen with a pocket lens.

[1] *Op. cit.*, p. 17, 1891.

The calcareous organisms were separated by decantations and carefully picked out by means of a camel hair brush and needle.

These when examined with the help of a microscope were found to consist principally of the tests of *Globigerina*, numerous sponge spicules and a number of skeletons of radiolaria of several types. Thus this sample can be called a Globigerina ooze.

Before the estimation of CO_2 the sample was dried in an air oven at a temperature between 105°C and 110°C.

On an average of 4 determinations the amount of CO_2 was found to be 21·37% which gives the percentage for $CaCO_3$ in the sample of ooze to be 48·57.

The insoluble residue is clayey. Phosphate was also found to be present in this sample.

Reference may be made in this connection to the curves published by Major Sewell[1] showing the percentage of calcium carbonate present in the bottom deposits at different depths and in different regions. A study of the curves shows that the result obtained in the present case agrees very nearly with the curve (*d*) of Murray and Renard obtained for open ocean deposits, as published by Major Sewell.

Specimen II.

Sounding No. 21, R. I. M. S. *Investigator*, 20th October, 1897.

This specimen was obtained from a depth of 1,430 fathoms; the latitude and the longitude of the spot are 6° 29′ 5″ and 78° 34′ 7″, respectively.

The sample looks like a green mud. In the dried condition it is dark grey in colour and fine grained. It is porous, light and smooth to the touch. Some gritty particles are perceptible when the substance is passed between the fingers. Some foraminifera are visible to the naked eye as white specks on the dark-coloured background. They can be clearly seen with the help of a pocket lens. The substance, when acted on by dilute hydrochloric acid, effervesces moderately leaving behind some insoluble residue of a dark green colour. The substance is not at all brittle but, on the other hand, is very sticky with high consistency. The streak is dirty grey. When a fragment is placed in a little water it does not, like the first sample, fall to

[1] Mem. Asiat. Soc. Beng., Vol. IX, No. II, p. 42, 1925.

pieces readily but takes a longer time. The moistened sample exhales, when breathed upon, a peculiar disagreeable odour. When dry it adheres to the tongue. If a fragment be placed upon a blowpipe flame it will fuse like most clays to a small globule. When one of the lumps is rubbed briskly with the back of a finger-nail or any hard smooth body the surface assumes a glazed appearance, a property peculiar to all varieties of clay. A small fragment can be moulded into any form between the fingers. Some *Globigerina* shells and several radiolarian remains could be detected amongst the calcareous organisms which are not so abundant as in the previous sample. These organisms were separated by fractional decantation prior to the analysis of the specimen.

The sample was thoroughly dried before the estimation of CO_2 and as a result of 4 estimations the amount of carbon dioxide was found to be on the average 13·32% giving 30·28 as the percentage of $CaCO_3$.

The insoluble residue which was of a dark green colour was washed to get rid of any chlorides adhering to it. It was then dried in an oven at 105°C to 110°C and finally subjected to a complete chemical analysis.

Under the microscope some green particles were found to be present with some amorphous clayey matter and quartz grains. The green minerals are supposed to be *Glauconite* the presence of which is confirmed by the results of the chemical analysis given below. The results of this analysis also show the presence of a high percentage of SiO_2 and this has been found to be due to quartz. These are all held together with some cementing material of a phosphatic nature, the presence of which was qualitatively determined in the portion of ooze soluble in HCl. The results of this analysis are given below :—

SiO_2 ...	... 55·43.
Al_2O_3 ...	... 1·59
Fe_2O_3 ...	... 5·22
CaO ...	... Trace
MgO ...	... 0·38
K_2O ...	... 4·16
Na_2O ...	... 0·24
Loss on ignition	... 31·89
	98·91

By working out the results of this analysis the molecular proportions obtained are as follows :—

SiO_2	... ...	923
$Al_2O_3+Fe_2O_3$	... ...	49
$CaO+MgO$	... ...	9
K_2O+Na_2O	... ...	47

Attention may be drawn in this connection to a paper recently published by Dr. Fermor [1] in which the constitution of Glauconite has been discussed. A reference to this paper shows that F. W. Clarke[2] regarded Glauconite as a metasilicate approaching more or less closely the formula of the type $(R_2O+RO).R_2O_3.4\ SiO_2+$ Aq. while according to Hallimond,[3] the formula of the mineral is $R_2O.\ 4(R_2O_3+RO).10SiO_2+5\frac{1}{2}H_2O$.

A consideration of the result obtained in the foregoing analysis shows that after making an allowance for some Al_2O_3 required for clay present in the sample, the proportion $(R_2O_3+RO):R_2O$ is 1 : 1. This result is different from that obtained by Clarke or Hallimond but shows that very likely Dr. Fermor's suggestion that R_2O_3, R_2O and RO are mutually replaceable, is the correct interpretation. As the sample contains free quartz it is not possible to say anything definitely about the molecules of SiO_2.

So we can conclude that the sample of ooze is a mixture of clay and Glauconite with some quartz grains and fragments of calcareous foraminifera shells. These are all held together by some cementing material of a phosphatic nature and the green colour of the mud is due to the presence of Glauconite.

Specimen III.

Station 612. R.I.M.S. *Investigator,* 26th April, 1914. The latitude and the longitude of the place are 9°41′00″ N. and 75°29′00″E., respectively. The depth from which this sample of green mud was obtained is about 725 fathoms.

The specimen is of a dark green colour in the wet condition, has got a very bad, disagreeable and penetrating smell and is very sticky.

[1] Records, Geological Survey of India, Vol. LVIII, pt. 3, p. 330 *et seq.*, 1925.

[2] The Data of Geo-chemistry, p. 521, and Monogr. U. S. Geol. Surv., Vol. 43, p. 243.

[3] Mineral Mag., Vol. XIX, pp. 330-333, 1922.

There are in the substance several small nodules which are not more than half an inch long. When dried in the hot-air oven the substance loses much of its disagreeable odour. It is very fine-grained and has got a very small amount of calcareous foraminiferal tests which can be seen easily as minute white specks on the dark green background. It is not smooth to the touch and is more or less sandy in appearance. When a fragment is placed in water it does not fall to pieces, showing that the grains are firmly held together. It is scratched by the finger nail and the streak is deep green. It is porous to a slight extent. When a little of the substance is passed between the fingers some gritty particles are perceptible. When dry it adheres to the tongue and when acted on by dilute HCl it effervesces feebly due to the low percentage of calcium carbonate and leaves behind some insoluble residue of a green colour. When the substance is under the microscope, in a dried and powdered state several fragments of foraminiferal tests together with free quartz grains and some amorphous clayey matter are visible, showing the sandy nature of the specimen. Some green particles are also found to be present.

The calcareous organisms which were, in this case, of small quantity were separated in the usual way of fractional decantation and some *Globigerina* shells were recognised. The nodules when broken open reveal some irregular cracks and holes in them. The specimen was found to contain phosphates.

Before the estimation of CO_2 the specimen was dried in an air oven between 105°C and 110°C.

On an average of 4 estimations the amount of CO_2 was found to be 7·48% which gives the result for $CaCO_3$ in the specimen to be 17·00%.

The insoluble residue was dried and subjected to a complete chemical analysis with the following result:—

SiO_2	... ...	50·00
Al_2O_3	... ...	2·26
Fe_2O_3	... ...	5·70
CaO	... ...	Trace
MgO	... ...	1·55
K_2O	... ...	7·50
Na_2O	... ...	0·64
Loss on ignition	...	31·34
		98·99

From this analysis the molecular proportions of the different constituents have been calculated and the following figures obtained :—

SiO_2 ...	...	833
$Al_2O_3 + Fe_2O_3$	...	58
$CaO + MgO$	...	38
$K_2O + Na_2O$	...	89

These figures show that the formula for Glauconite may be worked out exactly in the same way as in the case of the previous specimen by treating R_2O_3 and RO as mutually replacing one another. The formula for the mineral becomes $R_2O.(R_2O_3 + RO).4SiO_2 +$ Aq., making an allowance for some Al_2O_3 which will be required for the formation of the clay present in the specimen. It may be pointed out that the result obtained in this case confirms the conclusion arrived at before regarding the constitution of Glauconite, and that the presence of quartz grains in the specimen is responsible for the high SiO_2 percentage.

It may be concluded that the mud in question is an admixture of clay and Glauconite with some quartz grains and some fragments of calcareous foraminifera. These are all held together by some cementing material of a phosphatic nature.

The nature of the constituent minerals of the specimens (II and III) show that they were not deposited very far away from the land and the presence of quartz grains points out that they were derived from a continental land mass.

The green colour of the mud is mainly due to the presence of Glauconite. This mud mixed with Glauconite also contains a greenish amorphous matter which in part at least appears to be of an organic nature for it blackens on being heated on a platinum foil, leaving an ash coloured by oxide of iron. This accounts for the high figure for "Loss on ignition" as given above.

I am informed by Major Sewell that some Barium nodules have been found in the adjoining areas in the Lacca Sea. The nodules of the specimen under discussion were therefore subjected to chemical investigation but gave no reaction for Barium. Phosphates were found to be present. The constituent minerals of these nodules are of the same type as those found in the mud in which they are

embedded. From the above considerations it may be concluded that the nodules found in association with the mud have been formed by the agglomeration of clay, glauconite grains, phosphatic substances and some fragments of calcareous foraminiferal shells, through some process the nature of which is not quite clear.

POLYPORACEAE OF BENGAL, PART VIII.

By

S. R. Bose, D.Sc. F.R.S.E., F.L.S.

Professor of Botany,
Carmichael Medical College, Calcutta.

This paper is in continuation of previous works on systematic study of Bengal Polyporaceae, which I am bringing out in local journals, in series of twelve species each, with plates and detailed descriptions.*

Polyporus.

1. Polyporus cervino-gilvus, Jungh.
   =Polystictus zeylanicus, Berk. (of Ceylon).

Distribution and Habitat—In Java, and Polystictus zeylanicus in island of Ceylon, Bombay and Khandala: now collected from Kaptai forest Chandraghona, Chittagong, in November 1921, growing in imbricate manner, several together on dead tree on a hill.

Pileus—Almost dimidiate with a narrow point at the base, slightly leathery, very thin, about ½mm. thick, internally yellowish-brown, about 4×2 cm., some shorter, in the form of an arc of a circle.

* Polyporaceae of Bengal.

Part I Proceedings of the Indian Association for the Cultivation of Science Vol. IV Part IV. 1918.

Part II Report of the Indian Association for the Cultivation of Science and Proceedings of the Science Convention for the year 1918-1920.

Part III Bulletin of the Carmichael Medical Collge No. I—1920.

Part IV Bulletin of the Carmichael Medical College No. II—1921.

Part V Bulletin of the Carmichael Medical College No. III—1922.

Part VI Report of the Indian Association for the Cultivation of Science and Proceedings of the Science Convention for the year 1919—1922.

Part VII—Report of the Indian Association for the Cultivation of Science and Proceedings of the Science Convention for the year 1920-21—1923.

Upper Surface—Hairy, hairs a bit rough and projected, not zoned, colour yellowish-brown.

Hymenial Surface—Of the same colour as the upper one, pores angular, unequal, walls of the pores away from the margin being projected like teeth, context reduced to a thin basal layer.

Margin—Thin, fairly entire.

Spores—Brownish, oval, 4-5 × 3-4 μ.

Setae—None.

Deposits of excretion on hyphae.

2. Polyporus conchoides, Mont.

Distribution and Habitat—In Ceylon, Madagascar, South Africa, Brazil, Nicaragua, Cuba; now collected from Kaptai forest, Chandraghona, Chittagong, in November 1921, growing on trunk of dead tree on a hill, two or three joined together in imbricate manner.

Pileus—Sessile, substance hard and coriaceous in drying up, thickne s about 1 mm., internally white, about 4 × 3½ cm., in the form of an arc.

Upper Surface—Not hairy, pure white, not zoned, rugulose on account of tiny globular out-growths on the upper surface.

Hymenial Surface—Yellowish-white, pores very minute, uniform, round and very shallow, pore-surface reduced to a layer.

Margin—Slightly involute.

Spores—Hyaline, oval, few, 4 × 2 μ.

Setae—None.

3. Polyporus Thawaitesii, B & Br.

= Polystictus Gaudichaudii, Lev.

Distribution and Habitat—In Ceylon and Perak; now collected from Calcutta, Reserved Forests of Sunderbuns and Madras, in 1921 & 1922, growing in imbricate rings on timber and dead *Genwa* log (Excoecaria Agallocha) in Sunderbuns.

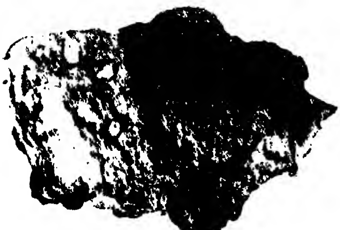
Pileus—With a short lateral stalk, some funnel-shaped, others lateral, fan-shaped, growing several together in imbricate manner forming a funnel, funnel varying from 4 to 7 cm. in length, substance stiffening in drying up, thin, about 1 mm. thick, internally white.

Upper Surface—Smooth, white, traversed by a number of narrow grey concentric zones.

Hymenial Surface—Of yellowish colour, pores unequal, elongated, depending on the plane of growth, pore walls a bit projected.

1a

1b



2a

2b

3a

3b

Margin—Distinctly involute.

Spores—Very few, faintly brown, round, diam. 4-6 μ.

Setae—None.

Pores here are smaller than the type specimen of *Polyporus Thawaitesii*.

4. Polyporus semilaccatus, Berk.

= Fomes semilaccatus, Berk.

Distribution & Habitat—In island of Ceylon, Malamon in the Philippines, and Japan; now collected from Reserved Forests of Sunderbuns and Madras in 1923 and 1921, growing on dead *Sundri* (Heritiera minor) log.

Pileus—Sessile, in one case with a lateral stalk of about 2 cm., applanate, substance very hard, about 12×6 cm., about $\frac{1}{2}$ cm. thick, internally of isabelline colour.

Upper Surface—Smooth, brownish, interspersed with dark concentric zones, not rugulose.

Hymenial Surface—Of pale brownish colour, pores minute, uniform and round, pore-tubes in stratified layers, usually in two layers, context a bit darker and woody.

Margin—A little wavy.

Spores—Not found.

Setae—None.

It is more of a Fomes than Polyporus. It was put in Fomes by Cooke. Lloyd says " It is hardly well-named although the darker blotches have a laccate effect ".

5. Polyporus luteo-umbrinus, Romel.

= Polyporus indicus, Massee.

Distribution and Habitat—Growing on trunks in Cuyaba and Matto Grosso in Brazil; now collected from Bakarganj District and Madras in 1921, growing on a dead fallen *Sundri* (Heritiera minor) tree.

Pileus—Dimidiate, substance hard, in the form of an arc, about 15×7 cm., about 1 cm. thick, internally brownish-yellow.

Upper Surface—Smooth, with a very thin black crust, faintly zoned.

Hymenial Surface—Brown, pores minute, uniform and round, pore-tubes in one layer, brown, context hard and yellow.

Margin—Fairly entire.

Spores—Many, brown, round, diameter 4-6 μ.

Setae—None.

6. Polyporus dissitus B & Br.

Distribution and Habitat—Growing on dead branches in Ceylon; now collected from Kadambagachi in 24 Perganas in 1921, growing laterally on dead date-palm trunk, forming step of a landing ghat.

Pileus—Sessile, substance hard in dried condition, about 6 × 2½ cm., about ½ c.m. thick, internally white.

Upper Surface—Smooth, white, not zoned, a bit rugulose or crumpled in dried condition.

Hymenial Surface—A bit darker, pores minute, round and uniform, pore tubes in one layer, context white and hard.

Margin—Involute.

Spores—Almost hyaline, some brownish, round, diam. 4-6 μ

Setae—None.

Bresadola and Lloyd consider that *Polyporus dissitus* is the same as *Polyporus adustus* of temperate climates.

7. Polyporus anthelminticus, Berk.

Distribution and Habitat—Growing at the base of bamboo trunk in Pegu, East India; now collected from Kadambagachi in 24-Perganas in September 1921, growing on dead root of bamboo.

Pileus—Stalked (with thick central stalk, about 1 cm. long), white, oval, substance soft, fleshy and brittle, the larger axis of the cap about 9 cm. and the shorter axis about 3½ cm., thickness of the cap about ½ cm., internally white.

Upper Surface—Smooth, whitish, or isabelline, crumpled in dried condition, not zoned.

Hymenial Surface—Of the same colour, pores broad, angular, and pressed close, pore-tubes in one layer, very short, about 1 mm. long, context soft and brittle.

Margin—Thinner, fairly entire.

Spores—Hyaline.

Setae—None.

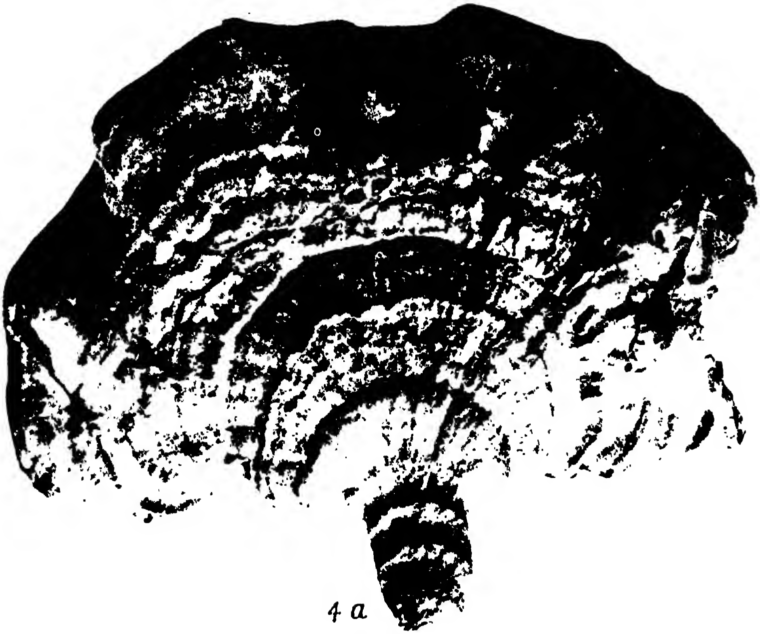
The specimen is supposed to have anthelmintic properties.

Polystictus.

8. Polystictus hypothejus, Kalchbr.

=Polystictus cryptominae, P. Henn.

Distribution and Habitat—Growing on trunks in Richmond River in Australia; now collected from Dooars Forests in Jalpaiguri



4 a

4 b

near Odlabari Railway Station in March 1921, growing in imbricate manner several joined together as lateral growth.

Pileus—Sessile, in the form of an arc, substance thin and coriaceous, thickness about 1 mm., internally white, undivided specimen about 3 × 2 cm.

Upper Surface—Smooth, faintly zoned, white, striated.

Hymenial Surface—Yellowish, pores angular, unequal, elongated, pore-walls a bit projected, context reduced to a thin basal layer bearing the pore-tubes.

Margin—Slightly wavy.

Spores—Not found.

Setae—None.

9. Polystictus vellereus, Berk.

= Polystictus fibula, Fr.

Distribution and Habitat—In island of New Ireland in Pacific Ocean, and Brisbane in Queensland; now collected from Kariah in 24-Perganas in August 1921, growing on root of dead bamboo.

Pileus—Sessile, in the form of an arc, substance a bit stiff in dried condition, and very thin, about ½ mm. thick, internally white, about 3 × 2 cm.

Upper Surface—Coated with soft minute hairs, hence velvety, greenish-yellow, concentrically zoned, soft to the touch.

Hymenial Surface—Yellow, pores uniform, round and medium, pore-tubes very short, yellow, context white.

Margin—Entire, with a thin marginal line.

Spores—Brownish, oval, $8 \times 4\mu$, some smaller, brownish, round, diam. 4μ.

Setae—None.

It has great similarity with the thin form of Trametes occidentalis.

10. Polystictus affinis, Nees.

Distribution and Habitat—Growing here and there irregularly on branches beneath grassy fields in the Philippine islands, Bombay —India, Nicobar and Aru, Java, Sumatra, Pegu, Perak, New South Wales in Australia, Ceylon, and Brazil; now collected from Mahisadal —Midnapur, in December 1920, growing on dead prostrate trunk of Terminalia Arjuna in imbricate manner, and in superposed layers.

Pileus—Almost sessile with a short lateral stalk, in the form of an arc, hard in dried condition, about 4 × 2 cm., about 2 mm. thick, internally yellowish white, thinning out towards the margin.

Upper Surface—With pubescent concentric zones, hence soft to the touch, of dark-bay colour, fainter towards the margin.

Hymenial Surface—Yellowish, pores very minute, uniform, and round, pore-tubes very short, less than ½ mm., context thick, about 2 mm., and stiff.

Margin—Entire, with a narrow pubescent zone on the lower surface, hence very soft and velvety to the touch.

Spores—Round with a thick wall, brown, diameter 8-10 μ.

Setae—None.

11. Polystictus steinheilianus, Berk. and Lev.

According to Dr. G. Bresadola of Italy (in Annales Mycologici XVIII) Polystictus connexus, Lev. = Polystictus steinheilianus, Berk. and Lev.

Distribution and Habitat—In Martinique, *Polystictuts connexus* was reportod from Rio-de-Jeniro growing on trunks; now collected from Puri-Orissa, in December 1921, growing on dead branch.

Pileus—Resupinate in the form of rounded and elongated patches, very thin and coriaceous.

Hymenial Surface—Yellowish, pores minute, uniform, and round, pore-tubes short, ½ mm., context reduced to a thin basal layer.

Margin—With a velvety outline.

Spores—Round, brown, diameter 8 μ.

Setae—None.

This specimen was kindly identified by Dr. G. Bresadola of Italy—but to me it seems a resupinate form of *Polystictus hirsutus*.

12. Poria membranicincta, Berk.

Distribution and Habitat—Growing on dead wood in island of Tasmania and Australia; now collected from Khulna in September 1921, growing on decomposed wood in the form of brownish thick elongated patches.

Pileus—Entirely resupinate, not easily separable from the surface of the wood on which it grows, colour white when first formed, but turning to brownish in course of time, patches at places extremely thin, and at others about 1 mm. thick.

5 a.

5 b

6 a

6 b

7

Hymenial Surface—On the exposed surface, pores unequal, some medium and round, others big and elongated depending on the plane of growth, pore-tubes extremely short.

Spores—Not found.

Setae—None.

BIBLIOGRAPHY.

1. Berkley, M. J.—Decades of Fungi in series in Hooker's Journal of Botany and Kew Garden Miscellany, Vols. II, III, IV and VI—1850-54.

2. Bijl, P. A. Vander—Polyporaceae of South Africa, Journal of Science, Vol. XVIII—1922-25.

4. Butler, E. J.—Fungi and Disease in Plants—1918.

4. Currey, F.—On Indian Fungi, Trans Linn. Soc., Series 2 Botany, Vol. I.

5. Hennings, P.—Fungi Indiae orientalis—Hedwigia Band XL—1901.

6. Lloyd, C. G.—Pamphlets on Stipitate Polyporoids and Fomes and Polypores and various Mycological Notes—1905-1925.

7. Massee, G.—Occasional notes on Indian Fungi in Kew Bulletins from 1901-1910.

8. McAlpine, D.—Australian Fungi—1895.

9. Murril, W. A.—Polyporaceae of North America—1902-1906.

10. Petch, T.—Revisions of Ceylon Fungi and Additions to Ceylon Fungi—Annals of the Royal Botanical Garden, Peradeniya, Ceylon—Vol. IV-VII—1907-1922.

11. Ramsbottom, J.—A Handbook of British Fungi—1923.

12. Rea, C.—British Basidiomycetae—1922.

13. Rostrup E.—Danish Fungi—Chapter on Polyporaceae—1913.

14. Saccardo, P.—Sylloge Fungorum, Vol. V—XXIII—1887-1925.

15. Streinz, W.—Nomenclator fungorum—1862.

16. Theissen, F.—Fungi aliquat Bombayensis—Annales Mycologici, Vol. IX, p, 155—1911.

17. Theissen, F.—Fungi of India—Journal Bom. Nat. Hist. Soc., Vol. XXII, No. I—1913.

8 a

8 b

9 a

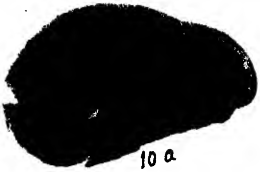
9 b

Polyporaceae of Bengal, Part VIII.

Explanation of the figures in the Plates :—

a=Upper surface. *b*=Hymenial surface.

1. Polyporus cervino-gilvus, Jungh.
2. Polyporus conchoides, Mont.
3. Polyporus Thawaitesii, B. and Br.
4. Polyporus semilaccatus, Berk.
5. Polyporus luteo-umbrinus, Romell.
6. Polyporus dissitus, B. and Br.
7. Polyporus anthelminticus, Berk.
8. Polystictus hypothejus, Kalchbr.
9. Polystictus villereus, Berk,
10. Polystictus affinis, Nees.
11. Polystictus steinheilianus, Berk. and Lev.
12. Poria membranicincta, Berk.



10 a

10 b

11 a

11 b

12

POLYPORACEAE OF BENGAL, PART IX.

By

S. R. Bose, D.Sc., F.R.S.E., F.L.S.,

Professor of Botany,

Carmichael Medical College, Calcutta.

This paper is in continuation of previous works on systematic study of Bengal Polyporaceae, which I am bringing out in local journals, in series of twelve species each, with plates and detailed descriptions.[1] For the collection of the specimens of the species Nos. 7 and 9 I am indebted to Major R. B. S. Sewell, I.M.S., Director, Zoological Survey of India; for those of the species No. 8 I am indebted to Drs. Hora and Chopra, Assistant Superintendents of the Indian Museum, Zoology Section; and for the specimens of the species No. 10 my grateful thanks are due to Drs. Hora and Rao of the Indian Museum, Zoology Section. Part of the expenses of the collection of some of the specimens (specially those of the species Nos. 3, 5 and 6) described in this paper was kindly met by a grant from the Royal Society of London.

1. Polyporous brumalis, Fr.

Distribution and Habitat :—Growing on trunks of Fagus, Quercus etc., common in Europe (Denmark), the United States of America, Middle and Upper Carolina, on ground in New-field, in Africa in mountainous forests of Baschberg near Somerset-East, Cape of Good

[1] Polyporaceae of Bengal.

Part I—Proceedings of the Indian Association for the Cultivation of Science, Vol. IV, Part IV, 1918.

Part II—Report of the Indian Association for the Cultivation of Science and Proceedings of the Science Convention for the year 1918-1920.

Part III—Bulletin of the Carmichael Medical College, No. I, 1920.

Part IV—Bulletin of the Carmichael Medical College, No. II, 1921.

Part V—Bulletin of the Carmichael Medical College, No. III, 1922.

Part VI—Report of the Indian Association for the Cultivation of Science and Proceedings of the Science Convention for the year 1919-1922.

Part VII—Report of the Indian Association for the Cultivation of Science and Proceedings of the Science Convention for the year 1920-21—1923.

Part VIII. Journal of the Department of Science, Calcutta University—1927.

Hope, in Queensland in Australia, on branches of Betula and Corylus in Europe ; now collected in August 1922 from Chilka Lake on the border of Orissa, growing on prostrate log.

Pileus—Centrally stalked, more or less circular, diameter about 2-6 cm., soft in growing condition, stiffening in dried state, about 2 mm. thick.

Stalk—2 to 6 cm. long, squamulose, thicker towards the base, of the same colour as the pileus.

Upper surface—Finely squamulose, dark-yellowish in colour, not zoned or rugulose.

Hymenial surface—Light-yellow, pores small and round, diameter about 110 μ, pore-tubes about 1 mm. long, slightly decurrent, yellowish, context soft and of lighter colour.

Margin—Involute in dried condition.

Spores—Not found.

2. Polyporus agariceus, Berk = Polyporus arcularius (Batsch), Fries = Favolus ciliaris, Mont.

Description and Habitat—In Ceylon Botanical Garden at Peradeniya, in Mussoorie, on trunks in Middle Europe and Southern Europe, Asiatic Siberia, Mount Langton, the United States of America and Central America, Victoria, New South Wales, and Queensland in Australia, Newzealand, on prostrate branches in Madagascar ; now collected from Berkuda Island, Chilka Lake, on the border of Orissa in July and September 1922, growing on ground as well as on a prostrate dead branch.

Pileus—Centrally stalked, circular and umbilicate, diameter about 2½ to 3½ cm., soft, very thin less than 1 mm.

Stalk—About 2-3 cm. long, almost uniform, of the same colour as the pileus.

Upper surface—Yellowish-brown, not zoned, at first squamulose, then becoming smooth, a bit rugulose in dried condition.

Hymeneal surface—Of light colour, pores big and hexagonal, pore-mouths rough and projected, pore-tubes about 1 mm. long, slightly decurrent, yellow, context reduced to a very thin membranous layer.

Margin—Slightly curved in, densely clothed with very minute short hairs, pores smaller towards the margin.

Spores—White, oval, about $8 \times 2-3$ μ.

Basidia—16×4 μ.

1a

1b

2 a

2 b

3 a

Cystidia—Present.

3. Polyporus Hookeri, Lloyd,

close to Polyporus scruposus.

Distribution and Habitat—Growing on trunks in India, Australia, and Mexico; now collected from Darjeeling in March, 1925, growing on the bark of a dead trunk, several together in imbricate manner in various rows.

Pileus—Sessile, in the form of an arc of a circle, hard and brittle in dried condition, about 6 × 3 cm., about 6 mm. thick, internally yellow.

Upper surface—Clothed with sharp stiff hairs, almost not zoned, rugulose towards the base on account of a number of very minute pustules, light-brown in colour, changing to blackish-brown on addition of a drop of ammonia.

Hymeneal surface—Deep-brown, pores small, round and uniform, pore-tubes about 2 mm. long, yellowish-brown, context deep-yellow, fibrous.

Margin—Thinner, usually fringed with a number of stiff yellowish-brown hairs which disappear with age.

Spores—Round, thick-walled, white, diameter about 6 μ.

Basidia—Very small, white.

Cystidia—Many, tapering at the apex, brown, thick-walled.

4. Polystictus sacer, Fr.

Distribution and Habitat—Growing on ground in Guinea and Java and South Africa; now collected from Hill tracts of Chittagong in June, 1926, growing on the ground from a big sclerotium (about the size of a cricket ball) at the base. Dr. E. J. Butler reported this species from the Hills of Assam where the sclerotium is locally known as "Tiger's milk" (*vide* Butler's book on '*Fungi and Disease in plants*,' p. 13, 1918).

Sclerotium—Very hard, with a thin laccate crust, upper surface rugulose, internally soft, white, and floccose.

Pileus—Centrally stalked, circular in the form of a spreading cup, slightly umbonate, stiff and coriaceous in dried condition, about 3 mm. thick, internally whitish.

Stalk—About 19cm. long, 8-10 mm. in cross-section, thinner towards the top, central surface laccate and brown, internally yellowish.

Upper surface—Smooth, distinctly striated, with a yellowish brown crust, concentrically zoned.

Hymeneal surface—Yellowish, pores small and round, pore-tubes about 2 mm. long, yellowish, changing to brownish on addition of a drop of strong ammonia, context thin, about 1 mm., yellowish.

Margin—Irregularly wavy in the form of the brim of a cup.

Spores—Not found.

5. Polystictus pterygodes, Fries.

Distribution and Habitat—On branches in Guinea and Singapur; now collected from Sylhet, Assam, in January, 1923.

Pileus—Dimidiate (almost sessile with a very short lateral stalk), in the form of an arc of a circle, stiff and coriaceous in dried condition, about $6\frac{1}{2} \times 4\frac{1}{2}$ cm., about 3 mm. thick, internally yellowish-white.

Upper surface—Smooth, yellowish-brown, shining, finely zoned.

Hymeneal surface—Yellowish-white, pores small and round, pore-tubes about 2 mm. long, yellowish, context thin, about 1 mm., whitish.

Margin—Thin with a soft yellowish line.

Spores—Not found.

Basidia—White, small, about 10×4 μ.

Cystidia—None.

Lloyd remarks: "It is perhaps simply a sessile condition of Polystictus xanthopus."

6. Polystictus perennis, Fries.

Distribution and Habitat—Growing everywhere on barren soil and also on trunks, in Europe (Denmark), near Minussinsk in Siberia, in the United States of America and Ceylon; now collected from Shillong, Assam, in September, 1926, growing from the ground.

Pileus—Centrally stalked, infundibuli form, diameter 2 to 4 cm., tough and coriaceous, about 1 mm. thick, internally yellowish-brown.

Stalk—About 2 cm., tawny, finely velvety, often bulbous at the base.

Upper surface—Velvety, cinnamon-coloured, distinctly zoned.

Hymeneal surface—Of lighter colour, sometimes whitish, changing to deep-brown on addition of a drop of strong ammonia, pores angular, not uniform, silvery, pore-tubes decurrent, yellowish, context thin, fibrous, and yellowish-brown.

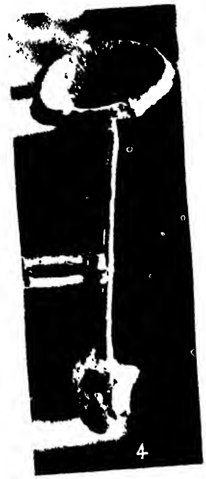
Margin—Entire.

Spores—White, oval, about 9×4 μ.

Basidia—White, about 16×6-8 μ.

Cystidia—None.

3 b



4

5 a

5 b

6 b

6 a

7. Polystictus velutinus, Fr.

Distribution and Habitat—On woody and leafy trunks in Europe (Denmark), near Minussinsk in Asiatic Siberia, Sarawak, Borneo, in the Philippine Islands, in Japan, in Middle Carolina, North America, Cuba in Central America, near Concepcion, Uruguay, San Carlos, Brazil, New Zealand, Java, Victoria, Queensland, and South Africa; now collected from Calcutta and Pashok (Darjeeling) at altitude of 2,500 ft. in 1925 and January, 1927, growing on dead wood of *Alnus nepalensis*, one above the other in imbricate manner.

Pileus—Almost sessile, in the form of an arc of a circle, tough and coriaceous in dried condition, about 3-5 × 2-2½ cm., about 1½ mm. thick, internally whitish to yellowish.

Upper surface—Minutely velvety, hairs very soft, white, becoming yellowish in time, very faintly zoned.

Hymeneal surface—Whitish, or yellowish, pores small and round, pore-tubes white to yellowish, very short, about 1 mm. long, context very thin, whitish, coriaceous.

Margin—Thin.

Spores—Not found.

Basidia—White, about 12-16 × 4 μ.

Setae—None.

It comes close to *Polystictus hirsutus*, Fr., from which it differs in having the upper surface very soft and velvety and very faintly zoned. In *Polystictus hirsutus* the upper surface is rough on account of very rigid hairs, and it is distinctly zoned.

8. Fomes annosus, Fries = Tramates radiciperda, Hartig.

Distribution and Habitat—On roots of stems, specially on cavities of trunks in Europe (Denmark), North America, and Cuba, on trunks of *Pinus silvestis* in Poland, Queensland (Australia); now collected from Shillong in September, 1926, growing at the base of the stumps of Pine trees as well as on Pine-wood paling. It usually causes disease ("heart-rot") of Coniferous trees acting as a root-parasite. W. G. Smith has recorded that the mycelium and the fruit body of this species are luminous.

Pileus—Almost sessile, sometimes resupinate, hard in dried condition, 5-10 × 4-7 cm., imbricate, of no regular shape, somewhat applanate, internally white or yellowish.

Upper surface—Bay-brown, not hairy, very rugulose with thin rigid crust, zoned.

Hymeneal surface—Yellowish, pores round, some angular, somewhat shining, pore-tubes yellowish, 2-5 mm. long, somewhat stratified, context whitish, soft, becoming hard in dried condition.

A number of holes are prominently seen in the hymeneal surface, because the fungus embeds pieces of sticks or dried leaves it may come across in course of its growth.

Margin—Much thinner, at first white.

Spores—Hyaline, subglobose, 6-7 × 6 μ.

Setae—None.

Fructifications are either subterranean (in case of resupinate ones), or they grow close to the ground always near the base of the trunk. Hiley has recorded in his book on "The Fungal diseases of the common Larch" (1919) that this particular habit of the fungus is due to its infection always spreading from the roots to the stems of the host plant and not *vice versa*.

9. Lenzites Malaccensis, Sacc. et Cub.

=Lenzites platyphylla, Cooke.

Distribution and Habitat—On rotten wood in Perak and Malacca Peninsula; now collected from Kulsi Range in Kamrup Division (Assam), Pashok near Darjeeling, and from North Burma Myitkyina District in 1925 and 1926, growing on prostrate trunks and stumps.

Pileus—Almost sessile with a short lateral stalk, stiff and leathery in dried condition, semicircular, in some cases almost circular, about 10-14 × 6 cm., about 7 mm. thick, thinning out towards the margin.

Upper surface—Smooth, marked with a number of tiny pustules, greyish-white, mostly not zoned.

Hymeneal surface—Broken up into a number of gills running divergently parallel, breadth of gills being a little less than 1 mm., and their depth about 6 mm., colour becoming yellowish in dried specimens, edge of gills very slightly toothed, context white, thin, soft, and floccose.

Margin—Much thinner and hence curved down in most cases.

Spores—Not found.

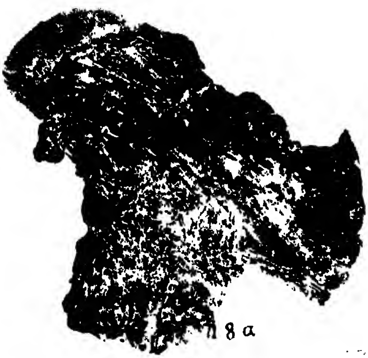
Setae—None.

10. Lenzites sepiaria, (Wulf) Fries.

Distribution and Habitat—On pine-wood, common in Europe (Denmark), near Minussinsk in Siberia, in the United States of

7a

7b



8a

8b

9a

America, in Victoria (Australia); now collected from Elephant Falls in Shillong in September, 1926, growing on palings of wooden bridge.

Pileus—Sessile, in some cases with a short lateral stalk, coriaceous, in the form of an arc, sometimes orbicular, sometimes extended longitudinally, about 2-7 × 1-3 c.m., about 5 mm. thick.

Upper surface—Minutely hairy, tomentose, finely pitted, yellow to brown in colour, in old specimens black, concentrically zoned.

Hymeneal surface—Broken up into a number of gills running parallel and partly anastomosing, yellowish to umber, turning to black on addition of a drop of strong ammonia, gills about 1½ mm. broad, and about 4 mm. deep, very firm, their edges partly toothed, context yellowish-brown, thin, and soft.

Margin—Thin with a yellowish line, entire.

Spores—Not found.

Setae—None.

11. Lenzites adusta, Massee.

=Lenzites Beckleri, Berk.

Distribution and Habitat—Growing on wood in Bengal; now collected from Sylhet, Assam, in January 1923, growing on dead tree.

Pileus—Almost sessile with a very short lateral stalk, arc-shaped, in most cases semicircular, about 3 mm. thick, internally pale-yellowish.

Upper surface—Smooth, dull-white, not zoned, the surface towards the base rough on account of the growth of small rounded pustules.

Hymeneal surface—Golden-yellow, gills running divergently parallel, golden-yellowish, narrow (breadth of gills about 1 mm.), crowded, and at places torn, context white, reduced to a thin basal layer.

Margin—Entire, involute.

Spores—Not found.

Setae—None.

12. Daedalea unicolor, (Bull.) Fries.

Distribution and Habitat—On trunks and on bark of woody and leafy trees, everywhere in Europe, near Minussinsk in Asiatic Siberia, in the United States of America, in Queensland and Victoria in Australia; now collected from Pashok close to Tista Bridge

(Darjeeling) in December 1926, and also from Kawnghka in North Shan States in Burma, in December, 1926, growing on prostrate trunks and stumps in imbricate manner.

Pileus—Dimidiate, in the form of an arc of a circle as well as of longitudinal extension, tough and coriaceous in dried condition, very thin, about 1 mm. thick, about 5-8 × 3½ cm., in dried condition soaking water like a piece of blotting paper.

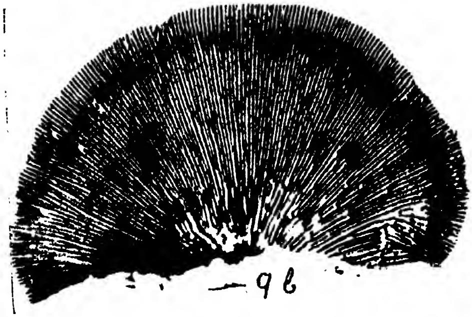
Upper surface—Strigose, hairs rather long and rough, yellowish-brown in fresh specimens, whitish-grey in very old specimens, concolorously zoned.

Hymeneal surface—Yellowish-brown, turning to deep brown on addition of a drop of ammonia, pores intricate and labyrinthiform, pore-mouths torn into teeth, at places resembling the hymeneal surface of a Hydnum, pore-tubes short, context very thin and yellowish.

Margin—Sometimes whitish.

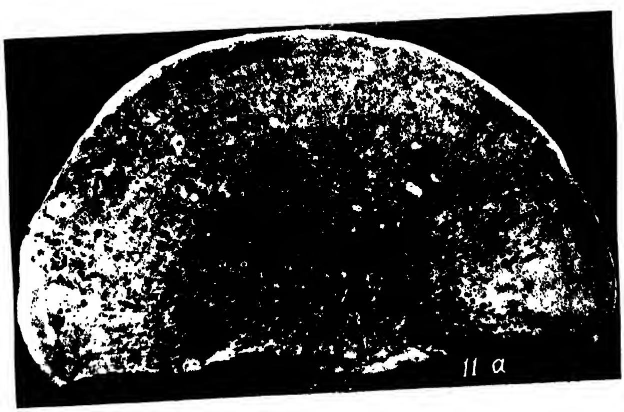
Spores—Not found.

Setae—None.


9b


10a


10b


11 a

BIBLIOGRAPHY.

1. Berkley, M. J.—Decades of Fungi in series in Hooker's Journal of Botany and Kew Garden Miscellany Vols. II, III, IV and VI—1850-54.

2. Bijl, P. A. van der—Polyporaceae of South Africa, South African Journal of Science, Vol. XVIII—1922-25.

3. Butler, E. J.—Fungi and Disease in Plants—1918.

4. Currey, F.—On Indian Fungi, Trans. Linn. Soc., Series 2 Botany, Vol. I.

5. Hennings, P.—Fungi Indiae Orientalis—Hedwigia Band XL—1901.

6. Hiley, W. E.—The Fungal Diseases of the Common Larch—1919.

7. Lloyd, C. G.—Pamphlets on Stipitate Polyporoids and Fomes and Polypores, and various Mycological Notes—1905-1925.

8. Massee, G.—Occasional notes on Indian Fungi in Kew Bulletins from 1901-1910.

9. McAlpine, D.—Australian Fungi—1895.

10. Murril, W. A.—Polyporaceae of North America—1902-1906.

11. Petch, T.—Revisions of Ceylon Fungi and Additions to Ceylon Fungi—Annals of the Royal Botanical Garden, Peradeniya, Ceylon—Vols. IV-VII—1907-1922.

12. Ramsbottom, J.—A Handbook of British Fungi—1923.

13. Rea, C.—British Basidiomycetae—1922.

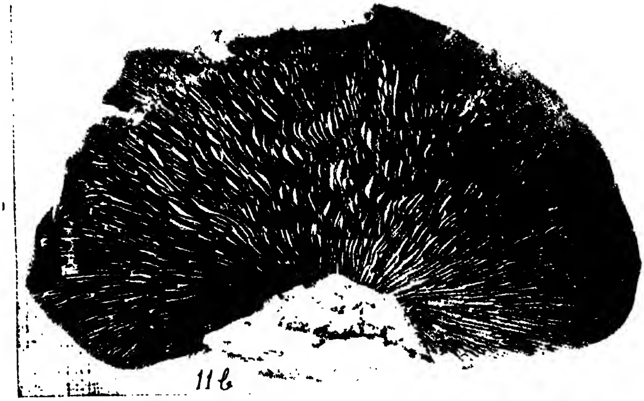
14. Rostrup E.—Danish Fungi—Chapter on Polyporaceae—1913.

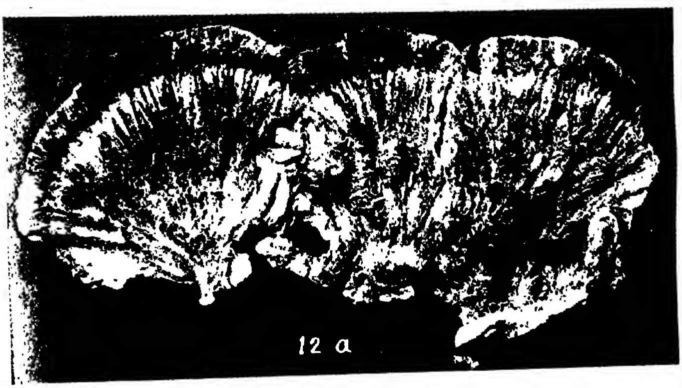
15. Saccardo, P.—Sylloge Fungorum, Vols. V-XXIII—1887-1925.

16. Streinz, W.—Nomenclator Fungorum—1862.

17. Theissen, F.—Fungi aliquat Bombayensis—Annales Mycologici, Vol. IX, pp. 155—1911.

18. Theissen, F.—Fungi of India,—Journal Bom. Nat. Hist. Soc., Vol. XXII, No. I—1913.


11 b


12 a


12 b

Polyporaceae of Bengal, Part IX.

Explanation of the figures in the Plates :—

a = Upper surface, *b* = Hymenial surface.

1. Polyporus brumalis, Fr.
2. Polyporus agariceus, Berk.
3. Polyporus Hookeri, Lloyd.
4. Polystictus sacer, Fr.
5. Polystictus pterygodes, Fr.
6. Polystictus perennis, Fr.
7. Polystictus velutinus, Fr.
8. Fomes annosus, Fr.
9. Lenzites Malaccensis, Sacc. et Cub.
10. Lenzites sepiaria, (Wulf.) Fr.
11. Lenzites adusta, Massee.
12. Daedalea unicolor, (Bull.) Fr.

MATHIEU FUNCTIONS

MATHIEU FUNCTIONS

BY

S. C. DHAR, D.Sc.

PREMCHAND ROYCHAND SCHOLAR, FORMERLY LECTURER, CALCUTTA UNIVERSITY ; PROFESSOR OF MATHEMATICS, COLLEGE OF SCIENCE, NAGPUR

CALCUTTA UNIVERSITY PRESS

1928

PRINTED AND PUBLISHED BY BHUPENDRALAL BANERJEE,
CALCUTTA UNIVERSITY PRESS, SENATE HOUSE, CALCUTTA.

Reg. No. 308 B J.—19-6-28—600.

PREFACE.

The subject of the Elliptic Cylinder Functions, though important, is yet very imperfectly developed and there still remain many things to be discovered before it will be brought in a line with other functions of the hypergeometric type.

The only book on the subject available at present is the admirable Treatise on Modern Analysis by Professors Whittaker and Watson, where the authors have devoted a chapter on Mathieu Functions. In the present book I have attempted to give a few methods by which the integrals of the first as well as the second kind can be found and their properties studied. It will be seen that the construction of the integrals is a difficult and tedious operation and I have given several alternative methods for the purpose. The best method of constructing the integrals and of studying their properties would, however, be by the help of recurrence-formulæ; but unfortunatel, no such formulæ have as yet been discovered. In their absence we are to rely upon other methods.

I have not treated the subject exhaustively, but have touched only those points which are not to be found elsewhere. In the end I have given an application of these functions to the problem of the scattering of electromagnetic waves by an elliptic cylinder and also by a thin blade.

I express my thanks to the authorities of the Calcutta University for undertaking its publication. My thanks are also due to the staff of the University Press for their valuable assistance.

College of Science, Nagpur,
March, 1927.

S. C. Dhar.

CONTENTS.

CHAPTER I.

Solutions of the first kind.

Mathieu's Differential Equation came to be studied in connection with the solutions of some physical problems and solutions of that equation under certain circumstances, came to be known as Mathieu Functions from the name of the discoverer, Emile Mathieu, who was the first to consider these functions. He was led to the discovery of these functions in solving the problem of the vibration of an elliptic membrane. For this reason, Mathieu Functions are also called "Elliptic Cylinder Functions," just as Legendre's or Bessel's Functions are also known as the Zonal or the Cylindrical Harmonics respectively.

As already alluded, these functions arise from the following equation of two dimentional wave-motion :—

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \ldots \qquad \ldots \quad (1)$$

Now, in order that the problem of the vibration of an elliptic membrane may be completely solved, it is necessary to use elliptic co-ordinates in place of rectangular ones. Let us suppose

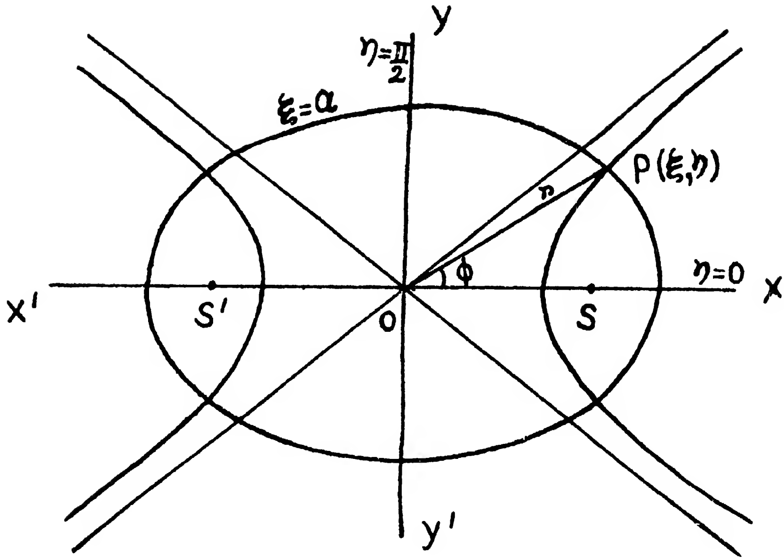
$$x=h\cosh\xi\cos\eta,\ y=h\sinh\xi\sin\eta.$$

Then, we have

$$\frac{x^2}{h^2\cosh^2\xi}+\frac{y^2}{h^2\sinh^2\xi}=1\,;\quad \frac{x^2}{h^2\cos^2\eta}+\frac{y^2}{h^2\sin^2\eta}=1. \qquad \ldots \quad (2)$$

The curves $\xi=$const. give a system of confocal ellipses and orthogonal with them, passes a system of confocal hyperbolas given by $\eta=$const. The foci of the confocal system are the points

$$x=\pm h,\quad y=0.$$



Thus, it is evident that through any point on the plane, two curves of the confocal system—one, a hyperbola and the other, an ellipse—must pass and it may conveniently and without any ambiguity, be denoted by (ξ, η). The range of values of ξ and η may be denoted as

$$\xi \geq 0, -\pi \leq \eta \leq +\pi.$$

Denoting the radius vector to the point P (ξ, η) by r, we have

$$r=h\sqrt{\cosh^2\xi-\sin^2\eta}=\frac{1}{2}h\sqrt{e^{2\xi}+2\cos\eta+e^{-2\xi}} \quad \dots \quad (3)$$

For large values of ξ, we can neglect $\sin^2\eta$ in comparison with $\cosh^2\xi$ and keep

$$r=h\cosh\xi=\frac{a}{2}e^{\xi}$$

Hence we get

$$x=r\cos\eta,\ y=r\sin\eta.$$

The line element in elliptic co-ordinates will be

$$ds^2 = h^2 (\cosh^2\xi - \cos^2\eta)(d\xi^2 + d\eta^2) = \rho^2 (d\xi^2 + d\eta^2), \quad \dots \quad (4)$$

where $\rho^2 = h^2 (\cosh^2\xi - \cos^2\eta)$

2. If we write

$$V = U(x, y) \cos pt,$$

the differential equation (1) transforms into :—

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{p^2}{c^2} U = 0, \quad \dots \quad \dots \quad \dots \quad (5)$$

which, on transformation into elliptic co-ordinates, becomes

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{h^2 p^2}{c^2} (\cosh^2\xi - \cos^2\eta) U = 0, \quad \dots \quad \dots \quad (6)$$

In the above differential equation, if we assume particular integral of the form

$$U = F(\xi) . G(\eta),$$

where $F(\xi)$ and $G(\eta)$ are exclusively functions of ξ and η respectively, we arrive at equations of the forms :—

$$\frac{d^2 G}{d\eta^2} - \left(\frac{h^2 p^2}{c^2} \cos^2\eta - a \right) G = 0, \quad \dots \quad \dots \quad (7)$$

$$\frac{d^2 F}{d\xi^2} + \left(\frac{h^2 p^2}{c^2} \cosh^2\xi - a \right) F = 0, \quad \dots \quad \dots \quad (8)$$

where a is an arbitrary constant.

Of these, the second comes from the first, when one puts '$i\xi$' for 'η.' Both, therefore, come from the equation

$$\frac{d^2 y}{dz^2} + (A + 16q \cos 2z) y = 0, \quad \dots \quad \dots \quad (9)$$

where $A = a + 16q$, and $32q = -\frac{h^2 p^2}{c^2}$,

which is the *standard form* for Mathieu's differential equation and under certain circumstances, particular solutions of it are called Mathieu Functions.

The physical problem which suggested Mathieu's equation, requires that its particular solutions must be periodic functions of "z" and hence periodic solutions of it have been considered by Mathieu. The solutions of the above differential equation are not in general periodic functions of "z"; but there is an infinite number of solutions which are periodic functions of "z"; just as solutions of Legendre's equation are not in general polynomials in "z," although there is an infinite number of them which are polynomials

in "z." It is, therefore, evident that for this, the arbitrary constant "A" must have certain *special values*.

EXISTENCE THEOREM.

3. The existence of an infinite number of periodic solutions of (9) corresponding to an infinite number of values of "A" can be seen thus :—

The periodic solutions of the equation (9) may assume either of the forms :—

$$\left.\begin{aligned}
&(i) \quad ce_{2r}(z, q) = a_{0,r} + \sum_{n=1}^{\infty} a_{n,r} \cos 2nz,\\
&(ii) \quad ce_{2r+1}(z, q) = \sum_{n=0}^{\infty} \beta_{n,r} \cos (2n+1)z,\\
&(iii) \quad se_{2r}(z, q) = \sum_{n=1}^{\infty} \gamma_{n,r} \sin 2nz,\\
&(iv) \quad se_{2r+1}(z, q) = \sum_{n=0}^{\infty} \delta_{n,r} \sin (2n+1)z.
\end{aligned}\right\} \quad \ldots \quad (10)$$

Substituting these in equation (9), we get the following recurrence formulae for calculating the co-efficients :—

$$\left.\begin{aligned}
&(i) \quad a_{1,r} = -\frac{A}{8q} a_{0,r},\\
&\qquad a_{n+1,r} = \frac{1}{8q}(4n^2 - A) a_{n,r} - a_{n-1,r},\\
&(ii) \quad \beta_{1,r} = \frac{1}{8q}\{(1-A) - 8q\}\beta_{0,r};\\
&\qquad \beta_{n+1,r} = \frac{1}{8q}\{(2n+1)^2 - A\}\beta_{n,r} - \beta_{n-1,r},\\
&(iii) \quad \gamma_{2,r} = \frac{1}{8q}(4-A)\gamma_{1,r};\\
&\qquad \gamma_{n+1,r} = \frac{1}{8q}\{(2n)^2 - A\}\gamma_{n,r} - \gamma_{n-1,r},\\
&(iv) \quad \delta_{1,r} = \frac{1}{8q}\{(1-A) + 8q\}\delta_{0,r};\\
&\qquad \delta_{n+1,r} = \frac{1}{8q}\{(2n+1)^2 - A\}\delta_{n,r} - \delta_{n-1,r}
\end{aligned}\right\} \quad \ldots \quad (11)$$

These series (10) converge only for determinate values of the parameter " A " of the differential equation and have for other values, no meaning. In fact for the convergence of the Fourier series (10), it is necessary but not sufficient that the co-efficients $\alpha_{n,r}$; $\beta_{n,r}$; $\gamma_{n,r}$ and $\delta_{n,r}$ shall vanish in the limit when " n " is indefinitely increased,

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} \alpha_{n,r}=0, \ \lim\limits_{n\to\infty} \beta_{n,r}=0 \\ \lim\limits_{n\to\infty} \gamma_{n,r}=0, \ \lim\limits_{n\to\infty} \delta_{n,r}=0 \end{array}\right\} \quad \ldots \quad (12)$$

for all values of r.

It will appear from the recurrence formulae (11) that these co-efficients $\alpha_{n,r}$, $\beta_{n,r}$, $\gamma_{n,r}$, and $\delta_{n,r}$ can be expressed as rational integral functions of " A " and the convergent conditions (12) will, therefore, furnish us with equations for determining the values of the parameter " A ". Heine* investigated these equations with the help of Sturms' Theorem and found that each of them has an infinite number of roots for " A ". These roots are all real, if q is real. He has further proved that for "n" very great

$$\lim \frac{\alpha_{n,r}}{\alpha_{n-1,r}}=0, \quad \lim \frac{\beta_{n,r}}{\beta_{n-1,r}}=0.$$

$$\lim \frac{\gamma_{n,r}}{\gamma_{n-1,r}}= \qquad \text{and} \quad \lim \frac{\delta_{n,r}}{\delta_{n-1,r}}=0.$$

By Cauchy's criteria of convergency, the convergence of the series (10) follows very easily. Thus the differential equation (9) gives an infinite number of periodic integrals corresponding to the infinite number of roots of "A". By their forms as given in (10) the solutions form into four classes. The integrals of the first and second classes are even and those of the other two are odd functions of " z ".

* Heine: Kugelfunctionen.

4. *For a numerical calculation of the periodic integrals one should* refer back to the method as used by Mathieu himself* (long before Heine) to get the series of integrals. These series are not obtained in the form of Fourier series as denoted in (10) but proceed according to powers of "q". Now granting the existence of periodic integrals as proved by Heine, one can get Mathieu's results in the following way :—

If we put

$$A = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \text{......etc.} \qquad (13)$$

and

$$y = f_0(z) + q f_1(z) + q^2 f_2(z) + \text{......etc.} \qquad (14)$$

where $f_0(z)$, $f_1(z)$, $f_2(z)$,...are functions of z only, then substituting these values of " A " and " y " in the differential equation (9) and equating the different powers of " q " to zero, we shall have to solve the following system of differential equations to determine $f_0(z)$, $f_1(z)$,......etc.,

$$\left.\begin{array}{l} f_0''(z) + a_0 f_0(z) = 0 \\ f_1''(z) + a_0 f_1(z) = -\{a_1 f_0(z) + 16 \cos 2z . f_0(z)\} \\ f_2''(z) + a_0 f_2(z) = -\{a_2 f_0(z) + a_1 f_1(z) + 16 \cos 2z . f_1(z)\} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ f_n''(z) + a_0 f_n(z) = -\{a_n f_0(z) + a_{n-1} f_1(z) + \cdots \quad \cdots \\ \cdots \qquad + a_1 f_{n-1}(z) + \cos 2z . f_{n-1}(z) \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \text{etc.} \qquad \text{etc.} \qquad \text{etc.} \end{array}\right\} \quad (15)$$

* E. L, Mathieu : Liouville's Journal (2) Vol. XIII (1868).

5. If we take $a_0 = m^2$, then $f_0(z)$ is either $\cos mz$ or $\sin mz$. Taking only one of these values of $f_0(z)$ and substituting in the second of the above equations, we shall have to solve the equation

$$f_1''(z) + m^2 f_1(z) = -\{a_1 \cos mz + 16 \cos 2z \cos mz\}$$

$$= -a_1 \cos mz - 8\cos (m+2)z - 8\cos (m-2)z$$

for finding the form of $f_1(z)$.

A solution of the above equation can easily be obtained by adding up the particular solutions of

$$\frac{d^2y}{dz^2} + m^2 y = -a_1 \cos mz,$$

$$\frac{d^2y}{dz^2} + m^2 y = -8 \cos(m+2)z,$$

$$\frac{d^2y}{dz^2} + m^2 y = -8 \cos(m-2)z.$$

They are $-a_1 z \sin mz/2m$, $2 \cos (m+2)z/(m+1)$, and

$-2 \cos (m-2)z/(m-1)$ respectively.

$$\text{Hence } f_1(z) = \frac{-a_1 z \sin mz}{2m} + \frac{2 \cos (m+2)z}{m+1} - \frac{2 \cos (m-2)z}{m-1}.$$

Since we require the solution to be periodic, 'a_1' must be taken $=0$, Therefore 'a_1'$=0$ and

$$f_1(z) = -\frac{2 \cos (m-2)z}{m-1} + \frac{2 \cos (m+2)z}{m+1}$$

Hence generally to determine the forms of $f_n(z)$, we need find the particular integrals of :—

$$y''+m^2y=-a_n f_0(z).$$

$$y''+m^2y=-a_{n-1}f_1(z)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$y''+m^2y=a_1 f_{n-1}(z).$$

$$y''+m^2y=16 \cos 2z.\ f_{n-1}(z),$$

where the forms of $f_0(z), f_1(z), \ldots$&c.... $f_{n-1}(z)$ are already found out in a series of cosines of multiples of "z" ; and add them up. The expressions will contain a non-periodic term like $z \sin mz$, whose coefficient must be equated to zero to give the value of "a_n". Hence for that value of "a_n", $f_n(z)$ will be periodic.

It will be seen that to determine the forms of $f_0(z), f_1(z), f_2(z), \ldots\ldots$ we shall constantly be required to find the particular solutions of equations of the types :—

$$\left.\begin{array}{ll} \text{(i)} & y''+m^2y=A \cos (m+a)z, \\ \text{(ii)} & y''+m^2y=A \sin (m+a)z, \\ \text{(iii)} & y''+m^2y=A \cos mz, \\ \text{(iv)} & y''+m^2y=A \sin mz. \end{array}\right\} \quad \ldots \quad \ldots \quad (16)$$

and their particular integrals are given by

$$\left.\begin{array}{ll} \text{(i)} & y=-A \cos (m+a)z/a(2m+a), \\ \text{(ii)} & y=-A \sin (m+a)z/a(2m+a), \\ \text{(iii)} & y=A\, z \sin mz/2m. \\ \text{(iv)} & y=-A\, z \cos mz/2m. \end{array}\right\} \quad \ldots \quad (17)$$

6. By adopting the above process, we get that for

$$A=m^2+\frac{32q^2}{m^2-1}-\frac{128(5m^2+7)q^4}{(m^2-1)^3(m^2-4)}-\ldots\&c., \quad \ldots \quad (18)$$

the integral is

$$\cos mz+q\left\{-\frac{2\cos(m-2)z}{m-1}+\frac{2\cos(m+2)z}{m+1}\right\}+q^2\left\{\frac{2\cos(m-4)z}{(m-1)(m-2)}\right.$$
$$\left.+\frac{2\cos(m+4)z}{(m+1)(m+2)}\right\}+\ldots\&c., \ldots (19)$$

Similarly, if we start with $f_0(z)=\sin mz$, by proceeding exactly as above we get that for

$$A=m^2+\frac{32q^2}{m^2-1}-\frac{128c(5m^2+7)q^4}{(m^2-1)^3(m^2-4)}-\ldots \text{etc.}, \quad \ldots \quad (20)$$

the solution is

$$\sin mz+q\left\{\frac{2\sin(m-2)z}{m-1}+\frac{2\sin(m+2)z}{m+1}\right\}+q^2\left\{\frac{2\sin(m-4)z}{(m-1)(m-2)}\right.$$
$$\left.+\frac{2\sin(m+4)z}{(m+1)(m+2)}\right\}+\ldots\&c., \quad \ldots \quad (21)$$

It would be observed that the forms of "A" as obtained above, although similar in both cases up to q^{m-1}, would differ from terms containing "q^m". Further, the solutions (19) and (21) reduce to $\cos mz$ and $\sin mz$, when $q=0$. They were denoted by Prof. Whittaker by the notations $ce_m(z, q)$ and $se_m(z, q)$ respectively. Thus, the *elliptic cylinder functions* will be denoted as

$$\left.\begin{array}{l} ce_0(z,q),\ ce_1(z,q),\ ce_2(z,q),\ \ldots\ ce_m(z,q),\ \ldots \\ se_1(z,q),\ se_2(z,q),\ \ldots\ se_m(z,q),\ \ldots \end{array}\right\} \quad \ldots \quad (22)$$

corresponding to the forms

$$\left.\begin{array}{l} 1,\ \cos z,\ \cos 2z,\ \ldots\ldots\ \cos mz,\ \ldots\ldots \\ \sin z,\ \sin 2z,\ \ldots\ldots\ \sin mz,\ \ldots\ldots \end{array}\right\}$$

to which they reduce when we put $q=0$.

7. These periodic solutions can be determined in various other ways, notable among them being the method by means of integral equations, employed by Whittaker*

A general form of homogeneous integral equations of Whittaker's type is obtained in art. 3, Chap III. There is, however, another class of integral equations of the second kind by which these functions can be constructed. We proceed to develop it.

Let it be required to obtain the general solutions of differential equations of the type,

$$\frac{d^2y}{dz^2}+Qy=Ry, \qquad \dots \qquad \dots \qquad \dots \quad (23)$$

where Q & R are functions of z.

Suppose y_1 (z) and y_2 (z) are any two solutions of

$$\frac{d^2y}{dz^2}+Qy=0 \; . \; . \; , \; . \; . \; . \qquad \dots \qquad \dots \quad (24)$$

Let us now assume that

$$y(z)=u_1\,(z)\;y_1\,(z)+u_2\,(z)\;y_2\,(z), \qquad \dots \qquad \dots \quad (25)$$

is the general solution of (23), where u_1, and u_2 are two arbitrary functions of z, whose forms are to be obtained subject to the condition

$$y_1\frac{du_1}{dz}+y_2\frac{du_2}{dz}=0. \qquad \dots \qquad \dots \qquad \dots \quad (26)$$

Now, substituting y (z) of (25) in (23) and simplifying by the help of the above relation, we obtain

$$\frac{du_1}{dz}\;\frac{dy_1}{dz}+\frac{du_2}{dz}\;\frac{dy_2}{dz}=\text{R}\;(u_1\,y_1+u_2\,y_2), \qquad \dots \qquad \dots \quad (27)$$

But from (26) we have

$$-\frac{\dfrac{du_1}{dz}}{y_2}=\frac{\dfrac{du_2}{dz}}{y_1}=Z, \text{ (say)}$$

* E. T. Whittaker : Fifth International Congress of Mathematicians, 1912, Vol.I.

then

$$\frac{du_1}{dz} = -Zy_2 \ \& \ \frac{du_2}{dz} = Zy_1$$

and the relation (27) becomes,

$$Z\left(y_1 \frac{dy_2}{dz} - y_2 \frac{dy_1}{dz}\right) = R(u_1 y_1 + u_2 y_2), \quad \dots \quad \dots \quad (28)$$

Now, y_1 & y_2 being solutions of (24), we have

$$\begin{vmatrix} y_1'' & y_1 \\ y_2'' & y_2 \end{vmatrix} = 0$$

which, on integration reduces to

$$y_1 \frac{dy_2}{dz} - y_2 \frac{dy_1}{dz} = C, \text{ an arbitrary constant.} \quad \dots \quad \dots \quad (29)$$

Hence we have

$$CZ = R(u_1 y_1 + u_2 y_2).$$

Therefore the solution of the equation (23) will be given by

$$y(z) = u_1 y_1 + u_2 y_2$$

when we take

$$Z = \frac{R}{C} y(z)$$

With this value of z, we obtain the values of u_1 & u_2, as

$$(a) \ \frac{du_1}{dz} = -\frac{R}{C} y(z).\, y_2, \quad \text{or} \quad u_1 = A - \frac{1}{C}\int_a^z R(t).\ y(t) y_2(t) dt$$

$$(b) \ \frac{du_2}{dz} = \frac{R}{C} y(z).\, y_1, \text{ or } u_2 = B + \frac{1}{C}\int_a^z R(t).\ y(t) y_1(t) dt.$$

where A, B, a are arbitrary constants.

Hence the solution (25) reduces to an integral equation of Volterra's type as :—

$$y(z)=(Ay_1+By_2)+\frac{1}{C}\int^{z} R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z)\end{vmatrix} . y(t)dt, \qquad \ldots \quad (30)$$

where A, B, a are arbitrary constants.*

8. On examining the relation (29), we find that the equation (24) cannot have two independent solutions which are both odd or both even. Hence of the two solutions of (24), one is odd and the other is even. Let $y_1(z)$ be the odd and $y_2(z)$, the even solution. Then if R be an even function, the integral equation

$$y(z)=By_2(z)+\frac{1}{C}\int_0^{z} R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z)\end{vmatrix} y(t)dt, \quad \ldots \quad (31)$$

will give *even* solutions of equation (23). This can be seen as follows :—

Writing $-z$ for z in (31), we get

$$y(-z)=By_2(-z)+\frac{1}{C}\int_0^{-z} R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(-z) & y_2(-z)\end{vmatrix} y(t)dt$$

$$=By_2(z)-\frac{1}{C}\int_0^{z} R(-t)\begin{vmatrix} y_1(-t) & y_2(-t) \\ y_1(-z) & y_2(-z)\end{vmatrix} y(-t)dt$$

$$=By_2(z)+\frac{1}{C}\int_0^{z} R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z)\end{vmatrix} y(-t)dt.$$

which is of the same form as relation (31). Therefore $y(z)$ and $y(-z)$ satisfy the same integral equation. Now as the solution of an integral equation of second kind is unique, we must have

$$y(-z)=y(z)$$

or in other words, the relation (31) will give *even* solutions of (23).

* A similar method is followed in a paper by Mr. H. J. Priestley, *Proc. Lond. Math. Soc.* Vol. 20, Series II.

Similarly, the odd solutions will be given by the integral equation,

$$y(z)=Ay_1(z)+\frac{1}{C}\int_0^z R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z) \end{vmatrix} y(t)dt. \qquad \dots \quad (32)$$

9. Let us now apply the above method to find solutions of Mathieu's equation (9), which can, for the present purpose, be written in the form

$$\frac{d^2y}{dz^2}+m^2y=-(a'+16q\cos 2z)\,y,$$

where $A=m^2+a'$, a' being a function of q.

The *even* solutions of the above equation will be evidently given, for different values of m and a', by the integral equation

$$y(z)=B\cos mz+\frac{1}{m}\int_0^z (a'+16q\cos 2t)\begin{vmatrix} \cos mz & \sin mz \\ \cos mt & \sin mt \end{vmatrix} y(t)dt,$$

$$=B\cos mz+\frac{1}{m}\int_0^z (a'+16q\cos 2t)\sin m(t-z)y(t)dt, \qquad \dots \quad (33)$$

[*N.B.*—It is necessary at this stage to mention that Liouville * obtained a similar integral equation as solutions of the homogeneous differential equation

$$\frac{d^2y}{dz^2}+[\rho^2-\sigma(z)]y=0,$$

to which it is needless to enter.]

Suppose

$$y(z)=A_0(z)+qA_1(z)+q^2A_2(z)+\dots\text{etc.},$$

where $A_0(z)$, $A_1(z)$, $A_2(z)$......etc., are periodic functions of z only and do not contain q, and suppose also

$$a'=a_1q+a_2q^2+\dots\text{etc.}.$$

* *Liouville's Journal*, Vol. 2.

where a_1, a_2,...... are numerical constants,

and

$$B=1+b_1q+b_2q^2+b_3q^3+\ldots\text{etc.},$$

where b_1, b_2,...... are also numerical constants.

On substituting in the integral equation, we find

$$A_0(z)+qA_1(z)+\ldots\ldots=(1+b_1q+b_2q^2+\ldots)\cos mz$$

$$+\frac{1}{m}\int_0^z\{(a_1q+a_2q^2+\ldots)+16q\cos 2t\}\times$$

$$\sin m(t-z).\{A_0(t)+qA_1(t)+\ldots\}dt.$$

Hence equating the co-efficients of successive powers of q, we get

(i) $A_0(z)=\cos mz$

(ii) $A_1(z)=b_1\cos mz+\dfrac{1}{m}\displaystyle\int_0^z(a_1+16\cos 2t)\sin m(t-z)A_0(t)dt,$

(iii) $A_2(z)=b_2\cos mz+\dfrac{a_2}{m}\displaystyle\int_0^z\sin m(t-z)A_0(t)dt$

$$+\frac{1}{m}\int_0^z(a_1+16\cos 2t)\sin m(t-z)A_1(t)dt,$$

... ...

... ...

The expressions $A_1(z)$, $A_2(z)$,......etc., will have to be obtained in succession from the above relation, subject to the condition that a_1, a_2,......must be such as will give A_1, A_2,......etc., the periodic forms. The values of b_1, b_2,...... will be obtained from the condition that the expressions for A_1, A_2,...... will not contain $\cos mz$ as a term in them.

Thus on integrating, we obtain :—

(a) $A_1(z)=\left(b_1+\dfrac{4}{m^2-1}\right)\cos mz-\dfrac{a_1}{2m}z\sin mz$

$$+\left\{\frac{2\cos(m+2)z}{m+1}-\frac{2\cos(m-2)z}{m-1}\right\}$$

$\therefore$ if $a_1=0$ and $b_1=-\dfrac{4}{m^2-1}$, then

$$A_1(z)=\frac{2\cos(m+2)z}{m+1}-\frac{2\cos(m-2)z}{m-1},$$

$$(b)\quad A_2(z)=\left\{b_2-\frac{4(m^2+2)}{(m^2-1^2)(m^2-2^2)}\right\}\cos mz+z\sin mz$$

$$\times\left\{\frac{-a_2}{2m}+\frac{16}{m(m^2-1)}\right\}+\left\{\frac{2\cos(m-4)z}{(m-1)(m-2)}+\frac{2\cos(m+4)z}{(m+1)(m+2)}\right\}$$

$$\therefore\ \text{if}\ a_2=\frac{32}{m^2-1},\quad\text{and}\quad b=\frac{4(m^2+2)}{(m^2-1^2)(m^2-2^2)},$$

then $$A_2(z)=\frac{2\cos(m+4)z}{(m+1)(m+2)}+\frac{2\cos(m-4)z}{(m-1)(m-2)}$$

and so on.

Thus, we see that when

$$A=m^2+\frac{32q^2}{m^2-1}+\ldots\text{etc.}$$

the solution as obtained by the integral equation is

$$\cos mz+q\left\{-\frac{2\cos(m-2)z}{m-1}+\frac{2\cos(m+2)z}{m+1}\right\}$$

$$+q^2\left\{\frac{2\cos(m-4)z}{(m-1)(m-2)}+\frac{2\cos(m+4)z}{(m+1)(m+2)}\right\}+\ldots$$

$$\ldots\qquad\ldots\qquad\ldots\ \text{etc.},$$

which we have denoted by $ce_m(z, q)$.

Therefore, when

$$a'=\frac{32q^2}{m^2-1}+\ldots\text{etc.}\quad\text{and}\quad B=1-\frac{4q}{m^2-1}+\ldots\text{etc.},$$

the integral equation (33) will give us the solution $ce_m(z, q)$.

10. We will conclude this by working out a particular case of the above; *i.e.*, when $m=1$. Here we obtain the forms of A_0, A_1... ...etc., as given below :—

(*i*) $A_0(z)=\cos z$,

(*ii*) $$A_1(z)=b_1 \cos z+\int_0^z (a_1+16 \cos 2t) \sin (t-z) \cos t\, dt$$

$$=(b_1-1) \cos z-(4+\tfrac{1}{2}a_1)z \sin z+\cos 3z$$

$\therefore \quad a_1=-8, \quad b_1=1, \quad$ and $\quad A_1(z)=\cos 3z$.

(*iii*) $$A_2(z)=b_2 \cos z+a_2\int_0^z \sin (t-z) \cos t\, dt$$

$$+\int_0^z (a_1+16 \cos 2t) \sin (t-z) \cos 3t\, dt$$

$$=(b_2+\tfrac{2}{3}) \cos z-(\tfrac{1}{2}a_2+4) z \sin z+\tfrac{1}{3} \cos 5z-\cos 3z$$

$\therefore$ if $\quad a_2=-8, \quad b_2=-\tfrac{2}{3}, \quad$ then $\quad A_2(z)=\tfrac{1}{3} \cos 5z-\cos 3z$,

(*iv*) $$A_3(z)=b_3 \cos z+a_3\int_0^z \sin (t-z) \cos t\, dt$$

$$+a_2\int_0^z \sin (t-z) \cos 3t\, dt$$

$$+\int_0^z (a_1+16 \cos 2t) \sin (t-z) A_2(t)dt$$

$$=(b_3+\tfrac{1}{18}) \cos z+\left(4-\frac{a_3}{2}\right) z \sin z$$

$$+\tfrac{1}{3} \cos 3z-\tfrac{4}{9} \cos 5z+\tfrac{1}{18} \cos 7z$$

$\therefore$ if $\quad a_3=8, \quad$ and $\quad b_3=-\tfrac{1}{18}$,

then $$A_3(z)=\tfrac{1}{3} \cos 3z-\tfrac{4}{9} \cos 5z+\tfrac{1}{18} \cos 7z,$$ and so on.

Hence when

$$A=1-8q-8q^2+8q^3+\ldots\text{etc.},$$

the solution of the integral equation is

$$\cos z+q\cos 3z+q^2(\tfrac{1}{3}\cos 5z-\cos 3z)$$

$$+q^3(\tfrac{1}{3}\cos 3z-\tfrac{4}{9}\cos 5z+\tfrac{1}{18}\cos 7z)+\ldots\text{etc.},$$

which is nothing but the solution $ce_1(z, q)$, the expression for B being

$$B=1+q-\tfrac{2}{3}q^2-\tfrac{1}{18}q^3+\ldots\text{etc.}$$

N.B.—The integral $ce_1(z, q)$ can also be obtained as the solution of the integral equation

$$y(z)=\cos z+\int_0^z(a'+16q\cos 2t)\sin(t-z)y(t)dt.$$

where $$a'=-8q-8q^2+8q^3+\ldots\text{etc.}$$

CHAPTER II.

Solutions of the Second Kind.

1. Solutions of the second kind of Mathieu's equation, corresponding to the solutions of the first kind as found out in the previous section, were first studied simultaneously both by Sieger[1] and Aichi[2] in connection with the solutions of a certain diffraction problem. They were, however systematically studied by Lindsay Ince[3] who gave us two methods for constructing the series of integrals. These solutions like the solutions of the first kind, are not, however, periodic. The existence of such solutions can be easily demonstrated by the well-known theorem of linear differential equation of the second order :—

Theorem. If $y=v$ be a particular integral of the differential equation

$$\frac{d^2y}{dz^2}+Qy=0, \qquad \dots \qquad \dots \qquad (1)$$

then the most general solution of above equation is given by

$$y=v(A+B\int\frac{1}{v^2}dz).$$

Therefore, the solution of the second kind corresponding to the solution $y=v$ of the first kind is given by

$$y=Bv\int\frac{1}{v^2}dz, \qquad \dots \qquad \dots \qquad (2)$$

where " B " is an arbitrary constant.

2. Mr. Ince has by using the formula (2) calculated out some of the simplest of the integrals. He has, in fact, calculated the

[1] Sieger : Annalen der Physik Bd. 27.

[2] Aichi : " Tokyô Sugaku-Buturigakkwai Kizi," 2nd ser., Vol. IV, No. 14.

[3] Lindsay Ince : Proc. Edin. Math. Soc., Vol. XXXIII.

integral corresponding to $ce_0(z, q)$. But this process is very laborious in practice and even then one cannot get as many terms as one may like to get.

His other method is comparatively easy but requires for its determination, a knowledge of the forms of the integrals, which is furnished by the formula (2). All these integrals can also be easily constructed by following, with a little modification, the methods[1] of arts. 4 and 5, Chapter I. We will proceed thus:—

If, for instance, we require to find the integral of the second kind corresponding to $ce_m(z, q)$, we will take the expression for "A" as given in (18) art. 6, Chapter I, and try to solve the set of equations (15) art. 4, Chapter I, after taking $f_0(z)$ to be $\sin mz$ instead of $\cos mz$. Or, in other words the problem is, therefore, to determine $f_1(z)$, $f_2(z)$,......by solving the following equations:—

$$\left.\begin{array}{l} f''_1(z)+m^2 f_1(z)=-\{a_1 f_0(z)+16\cos 2z\, f_0(z)\}, \\ f''_2(z)+m^2 f_2(z)=-\{a_2 f_0(z)+a_1 f_1(z)+16\cos 2z\, f_1(z)\}, \\ \cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots \\ \cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots \end{array}\right\} \quad \ldots \quad (3)$$

where $f_0(z)=\sin mz$ and $a_1, a_2, a_3, \ldots$ which are the co-efficients of the different powers of q, are known from (18) art. 6, Chapter I.

3. By following the procedure of art. 5, it is easy to find that the second integral corresponding to the solution $ce_m(z, q)$ is given by

$$Aq^m z.ce_m(z, q)+\sin mz+q\left\{-\frac{2\sin(m-2)z}{m-1}+\frac{2\sin(m+2)z}{m+1}\right\}$$
$$+\ldots\ldots\text{etc.} \qquad \ldots \quad (4)$$

(A being a function of q.)

which, following the notation suggested by Whittaker, may be denoted by $in_m(z, q)$. Similarly, the integral corresponding to

[1] Dhar: "On Elliptic Cylinder Functions of the Second Kind," *American Jour. of Mathematics*, Vol XLV, No. 3.

$se_m(z, q)$ can also be obtained by taking $f_0(z) = \cos mz$ and also using the values of $a_1, a_2, a_3 \ldots\ldots$etc. from (20) art. 6. It is denoted by the notation $j\eta_m(z, q)$ and is given by

$$Bq^m z.\, se_m(z, q) + \cos mz + q \left\{ -\frac{2\cos(m-2)z}{m-1} + \frac{2\cos(m+2)z}{m+1} \right\}$$

$$+ \ldots\ldots \text{etc.} \qquad \ldots \quad (5)$$

where B is a certain function of " q."

Here also as in arts. 4 and 5, Chapter I, to solve the set of equations (3) we shall not only require the particular integrals (17) of equations of the types (16) Chapter I, but the particular integrals of the following types will also be much in use :—

$$\left.\begin{array}{ll} (1) & y'' + m^2 y = Az \cos az \\ (2) & y'' + m^2 y = Bz \sin az \end{array}\right\} \qquad \ldots \qquad \ldots \quad (6)$$

whose particular integrals are respectively

$$\left.\begin{array}{ll} (1) & y = \dfrac{A}{m^2 - a^2} z \cos az + \dfrac{2Aa}{(m^2 - a^2)^2} \sin az, \\[2ex] (2) & y = \dfrac{B}{m^2 - a^2} z \sin az - \dfrac{2Ba}{(m^2 - a^2)^2} \cos az. \end{array}\right\} \quad (7)$$

Thus following the above processes, all the integrals corresponding to the integrals of first kind as denoted in (22) Chapter I, can be obtained and they will be denoted as

$$\left.\begin{array}{l} i\eta_0(z, q), \quad i\eta_1(z, q), \quad i\eta_2(z, q), \ldots\ldots i\eta_m(z, q), \ldots\ldots \\ \quad j\eta_1(z, q), \quad j\eta_2(z, q), \ldots\ldots j\eta_m(z, q), \ldots\ldots \end{array}\right\} \quad \ldots \quad (8)$$

4. We will illustrate the processes of arts. 2 and 3 of the present Chapter, by working out a few particular cases. Suppose we wish to find the integral which corresponds to $ce_1(z, q)$. The particular value of " A " for this is given by

$$A = 1 - 8q - 8q^2 + 8q^3 - \tfrac{8}{3}q^4 - \tfrac{88}{9}q^5 + \ldots \text{etc.} \qquad \ldots \quad (9)$$

Hence to find $i\eta_1(z, q)$, we shall have to solve the equations :—

$$\left.\begin{array}{l}
(i)\ f''_1(z)+f_1(z)=8\sin z-16\cos 2z.\sin z.\\
(ii)\ f''_2(z)+f_2(z)=8\sin z+8f_1(z)-16\cos 2z f_1(z).\\
(iii)\ f''_3(z)+f_3(z)=-8\sin z+8f_1(z)+8f_2(z)-16\cos 2z f_2(z),\\
\dots\quad\dots\quad\dots\quad\dots\quad\dots\\
\dots\quad\dots\quad\dots\quad\dots\quad\dots
\end{array}\right\} \quad (10)$$

(a) Now, the solution of the equation (i), (10) is obtained by adding up the particular integrals of

$$y''(z)+y(z)=16\sin z,\quad y''(z)+y(z)=-8\sin 3z,$$

which by the help of (17) and (18), Chapter I, is given by

$$f_1(z)=-8z\cos z+\sin 3z$$

(b) To find $f_2(z)$, we find the particular integrals of the following equations :—

$$y''(z)+y(z)=64z\cos 3z.\qquad y''(z)+y(z)=8\sin 3z.$$

$$\text{and}\quad y''(z)+y(z)=-8\sin 5z$$

and add them up.

We get

$$f_2(z)=-8z\cos 3z+5\sin 3z+\tfrac{1}{3}\sin 5z.$$

(c) To find $f_3(z)$, we see that the expression on the right-hand side of (iii) art. 4, is

$$-48\sin z+\tfrac{136}{3}\sin 3z-\tfrac{112}{3}\sin 5z-\tfrac{8}{3}\sin 7z-16z\cos 3z-64z\cos 5z.$$

and, therefore, the form of $f_3(z)$ is obtained by adding up the integrals of

$$y''(z)+y(z)=-48\sin z.\ \dots\dots\dots\ y''(z)+y(z)=-64z\cos 5z.$$

and hence we have

$$f_3(z)=24z\cos z+8z\cos 3z-\tfrac{8}{3}z\cos 5z+\tfrac{1}{18}\sin 7z+\tfrac{8}{3}\sin 5z-\tfrac{26}{3}\sin 3z.$$

Proceeding thus, we can get to any term we like to have and hence $i\eta_1\,(z,q)$ is given by

$$-8q\,(1-3q^2+\ldots)\,z\{\cos z+q\cos 3z+q^2\,(-\cos 3z+\tfrac{1}{3}\cos 5z)+\ldots\}$$

$$+\sin z+q\sin 3z+q^2\,(\tfrac{1}{3}\sin 5z+5\sin 3z)+q^3\,(\tfrac{1}{18}\sin 7z+$$

$$\tfrac{5}{3}\sin 5z-\tfrac{35}{8}\sin 3z)+\ldots\ldots\text{etc.}$$

Another Method of constructing the solutions of the Second Kind.[1]

5. The methods given above and also employed by Mr. Lindsay Ince,[2] are not suitable for studying the convergence of the series but what is given below, while allowing us to construct easily the series of integrals, is also suitable for the consideration of their convergency. This latter method follows lines similar to that employed by Frobenius [3] in solving linear differential equations and also similar to that employed by Prof. Whittakar and Watson [4] for constructing integrals of the first kind.

Let us investigate the solution $i\eta_m\,(z,q)$ corresponding to the solution $ce_m\,(z,q)$ for which the value of " Λ " is given by

$$\Lambda=m^2+\frac{32q^2}{m^2-1}-\frac{128\,(5m^2+7)q^4}{(m^2-1)^3(m^2-4)}-\ldots\ldots\text{etc.} \qquad \ldots \qquad (11)$$

which we will denote for our purpose, as

$$\Lambda=m^2+a_1q+a_2q^2+a_3q^3+\ldots\ldots\ldots\text{etc.} \qquad \ldots \qquad (12)$$

where $\qquad a_1=0,\; a_2=\dfrac{32}{m^2-1},\;$ etc.

If we put in Mathieu's differential equation

$$\Lambda=m^2+8p$$

it will become

$$\frac{d^2y}{dz^2}+m^2y=-8\,(p+2q\cos 2z)y.$$

[1] Dhar : " *Tôhoku Math. Jour.*," Vol. 19 (1921).

[2] Lindsay Ince : *Loc. cit.*, Vol. XXXIII.

[3] Frobenius, Crelle's Jour., Vol. LXXVI.

[4] Whittaker & Watson : Modern Analysis, pp. 413-415.

6. When "p" & "q" are neglected, the solutions of the equation, are given by

$$y=\cos mz, \text{ and } y=\sin mz.$$

If we proceed with $y=\cos mz$, it will enable us to construct the solution $ce_m(z,q)$ [1] and so we proceed with $y=\sin mz$. Let us put $\sin mz=U_0(z)$. Then to obtain a closer approximation, we write $-8(p+2q\cos 2z)\,U_0(z)$ as a series of sines of multiples of "z" in the form

$$-8\{q\sin(m-2)z+p\sin mz+q\sin(m+2)z\}$$

which we will denote by $V_1(z)$.

Then instead of solving the differential equation

$$\frac{d^2y}{dz^2}+m^2y=V_1(z),$$

we will solve the equation

$$\frac{d^2y}{dz^2}+m^2y=W_1(z). \qquad \ldots \qquad \ldots \qquad (13)$$

where, $\qquad W_1(z)=V_1(z)+(8p-a_1q)\sin mz.$

Its integral, which we will denote by $U_1(z)$ is given by

$$U_1(z)=\frac{-2q\sin(m-2)z}{m-1}+\frac{2q\sin(m+2)z}{m-1}+\frac{a_1qz\cos mz}{2m} \qquad (14)$$

7. To obtain a still closer approximation, we will express $-8(p+2q\cos 2z)\,U_1(z)$ as a series of sines of multiples of "z" which we will denote by $V_2(z)$, *viz.*,

$$V_2(z)=\frac{16q^2\sin(m-4)z}{m-1}+\frac{16pq}{m^2-1}\sin(m-2)z$$

$$+\frac{32q^2}{m^2-1}\sin mz-\frac{16pq}{m+1}\sin(m+2)z-\frac{16q^2}{m+1}\sin(m+4)z$$

[1] Modern Analysis, p. 413.

$$-\frac{8q^2a_1}{2m}\,z\cos(m-2)z-\frac{8pqa_1}{2m}\,z\cos mz$$

$$-\frac{8q^2a_1}{2m}\,z\cos(m+2)z, \qquad \ldots \qquad \ldots \qquad (15)$$

Here again we solve the equation,

$$\frac{d^2y}{dz^2}+m^2y=W_2\,(z),$$

where $$W_2\,(z)=V_2\,(z)-aq^2\sin mz+\lambda_2 z\cos mz$$

and λ_2 has been determined in such a way that $W_2\,(z)$ does not involve "$z\cos mz$". [1]

Suppose that $U_2\,(z)$ is the integral of the above equation. We will now proceed exactly with $U_2\,(z)$ as we have done with $U_1\,(z)$, and obtain the integral $U_3\,(z)$.

8. Continuing thus, we get the series of integrals $U_0\,(z)$, $U_1\,(z)$, $U_2\,(z)$,......$U_n\,(z)$...... etc. of the differential equations,

$$\left.\begin{array}{l}\dfrac{d^2y}{dz^2}+m^2y=0,\\[2ex] \dfrac{d^2y}{dz^2}+m^2y=W_1\,(z),\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \dfrac{d^2y}{dz^2}+m^2y=W_n(z),\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{array}\right\} \qquad \ldots \qquad \ldots \qquad (16)$$

respectively, where

$$\left.\begin{array}{l}W_n(z)=V_n\,(z)-a_nq^n\sin mz+\lambda_n\,z\cos mz\ldots\ldots(n\geq 2)\\ V_n\,(z)=-8(p+2q\cos 2z)\,U'_{n-1}\,(z)\ldots\ldots(n\geq 1)\end{array}\right\} \qquad (17)$$

[1] It will be foundt hat $z\cos mz$ first appears in $U_m\,(z)$, *i.e.*, in $V_{m+1}\,(z)$ in this case.

Therefore, from (16) and (17) we have

$$\left(\frac{d^2}{dz^2}+m^2\right)\sum_{n=0}^{\infty} U_n(z)=\sum_{n=1}^{\infty} W_n(z)=\sum_{n=1}^{\infty} V_n(z)$$

$$+\left(8p-\sum_{n=1}^{\infty} a_n q^n\right)\sin mz+\sum_{n=2}^{\infty} \lambda_n z\cos mz,$$

or $$\left\{\frac{d^2}{dz^2}+(A+16q\cos 2z)\right\}\sum_{n=0}^{\infty} U_n(z)=\left(8p-\sum_{n=1}^{\infty} a_n q^n\right)\sin mz$$

$$+\sum_{n=2}^{\infty} \lambda_n z\cos mz.$$

But we have from (12)

$$8p=\sum_{n=1}^{\infty} a_n q^n,$$

and it will also be found that $\sum_{n=2}^{\infty}\lambda_n$ also vanishes for the above value of "p." Hence $\sum_{n=0}^{\infty} U_n(z)$ will be the solution of the second kind, if the series be uniformly convergent throughout a two-dimensional region of the z plane. We will, therefore, have

$$in_m(z,q)=\sum_{n=0}^{\infty} U_n(z)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

9. We proceed to show that $\sum_{n=2}^{\infty}\lambda_n$ actually vanishes by working out a few particular cases. Suppose we construct the integral which corresponds to $ce_1(z,q)$. The particular value of the arbitrary constant "A" for which $ce_1(z,q)$ is obtained, is given by

$$A=1-8q-8q^2+8q^3-\tfrac{8}{3}q^4-\tfrac{?}{?}q^5-\ldots\text{etc} \qquad (19)$$

If we write

$$A=1+8p,$$

Mathieu's differential equation reduces to

$$\frac{d^2y}{dz^2}+y=-8(p+2q\cos 2z)y.$$

In this case $U_0(z)$ is evidently $\sin z$.

(*a*) To get $U_1(z)$, we express $-8(p+2q\cos 2z)\,U_0(z)$ in a series of sines of the form

$$-8(p-q)\sin z-8q\sin 3z=V_1(z),\ (\text{say})$$

and solve the equation

$$\frac{d^2y}{dz^2}+y=W_1(z).$$

where $W_1(z)=V_1(z)+8(p+q)\sin z$, since there is no term as $z\cos z$ contained in $V_1(z)$.

The integral of the above equation is found by the help of arts. 5, 16, Chapter I, and arts. 8 and 7, Chapter II, to be

$$U_1(z)=-8qz\cos z+q\sin 3z.$$

(*b*) Again, we express $-8(p+2q\cos 2z)\,U_1(z)$ in a series of sines as

$64q(p+q)z\cos z+64q^2 z\cos 3z-8pq\sin 3z-8q^2\sin 5z-8q^2\sin z$, which we denote by $V_2(z)$.

Then we shall have to find an integral of the equation

$$\frac{d^2y}{dz^2}+y=W_2(z),$$

where $W_2(z)=V_2(z)-64q(p+q)z\cos z+8q^2\sin z$.

(Here $W_2(z)$ is made independent of $z\cos z$). On simplification, we get

$$W_2(z)=64q^2 z\cos 3z-8pq\sin 3z-8q^2\sin 5z.$$

The integral of the above equation is given by

$$U_2(z)=-8q^2z\cos 3z+q(6q+p)\sin 3z+\tfrac{1}{3}q^2\sin 5z.$$

(*c*) Again, since

$$V_3(z)=-8(p+2q\cos 2z)\,U_2(z)$$

$$=64q^3z\cos 5z+64pq^2z\cos 3z+64q^3z\cos z$$

$$-8q^2(6q+p)\sin z-\{8pq(6q+p)+\tfrac{8}{3}q^3\}\sin 3z$$

$$-8\{q^2(6q+p)+\tfrac{1}{3}pq^2\}\sin 5z-\tfrac{8}{3}q^3\sin 7z.$$

We shall have

$W_3(z) = V_3(z) - 64q^3 z \cos z - 8q^3 \sin z$, so that $W_3(z)$ may not contain $z \cos z$.

And solving the differential equation

$$\frac{d^2 y}{dz^2} + y = W_3(z).$$

we find

$$U_3(z) = -\tfrac{5}{3}q^3 z \cos 5z - 8pq^2 z \cos 3z + 4q^2(7q+p)z \cos z$$

$$+ q(v^2 + 12pq + \tfrac{1}{3}q^2) \sin 3z + \tfrac{4}{9}q^2(p+7q) \sin 5z + \tfrac{1}{18}q^3 \sin 7z.$$

(*d*) Again $V_4(z) = -8(p + 2q \cos 2z)\, U_3(z)$ and expressing it in a series of sines, we find that the term which contains $z \cos z$ as a factor is

$$-32q^2(p^2 + 6pq + 7q^2)\, z \cos z,$$

and the term which contains $\sin z$ is

$$-8q^2(p^2 + 12pq + \tfrac{4}{3}q^2) \sin z.$$

Hence we put $W_4(z)$ as

$$W_4(z) = V_4(z) + 32q^2(p^2 + 6pq + 7q^2) z \cos z + \tfrac{8}{3}q^4 \sin z,$$

$W_4(z)$ is thus made independent of $z \cos z$.

We can now find the integral $U_4(z)$.

Proceeding thus, we find in succession the U's.

10. In this case $\sum\limits_{n=2}^{\infty} \lambda_n = -64q(p+q) - 64q^3 + 32q^2(p^2 + 6pq + 7q^2) +$

....and if by (19) we substitute

$$8p = -8q - 8q^2 + 8q^3 - \tfrac{8}{3}q^4 - \ldots\ldots\ldots \text{etc.}$$

then neglecting higher powers of "p" than the fourth, it will be seen that $\sum\limits_{n=2} \lambda_n$ vanishes.

Also, if in the expression $U_0(z)+U_1(z)+U_2(z)+U_3(z)+\ldots\ldots$ as obtained above, we substitute for "$8p$" in terms of the "q's," we obtain (on arranging)

$$-8q(1-3q^2+\ldots)z\{\cos z+q\cos 3z+q^2(-\cos 3z+\tfrac{1}{3}\cos 5z)+$$

$$\ldots\ldots\}+\sin z+q\sin 3z+q^2(\tfrac{1}{3}\sin 5z+5\sin 3z)$$

$$+q^3(\tfrac{1}{18}\sin 7z+\tfrac{8}{3}\sin 5z-\tfrac{35}{3}\sin 3z)+\ldots\ldots\text{etc.}$$

This is, therefore, a solution of Mathieu's differential equation, corresponding to the value of A given by (19) § 9, Chap. II, and is the solution of the second kind denoted by $i\eta_1\ (z,\ q)$.

11. Similarly, if we proceed to construct the integral $i\eta_2\ (z,\ q)$, corresponding to the value of "A" for $ce_2\ (z,\ q)$, *viz.*,

$$A=4+\frac{80}{3}q^2-\frac{6104}{27}q^4+\ldots\ldots\text{etc.}$$

We find

$$U_0\ (z)=\sin 2z,$$

$$U_1(z)=\tfrac{2}{3}q\sin 4z,$$

$$U_2(z)=\tfrac{1}{6}q^2\sin 6z+\tfrac{1}{3}pq\sin 4z+8q^2z\cos 2z,$$

$$U_3(z)=\tfrac{1}{45}q^3\sin 8z+\tfrac{11}{72}pq^2\sin 6z+\tfrac{1}{9}q(\tfrac{8}{3}p^2-3kq)\sin 4z$$

$$-16q^3z+\tfrac{8}{3}pq^2z\cos 2z+\tfrac{16}{3}q^3z\cos 4z$$

....

....

and

$$\Sigma\lambda_n=64pq^2+\left(\frac{64}{9}p^2q^2-256q^4+\frac{128}{3}q^2\right)+\ldots\text{etc.}$$

On substituting the expression for "$8p$" in terms of "q" and neglecting higher power of "q" than the fourth, it will be seen that $\Sigma\lambda_n$ vanishes.

Hence

$$\sum_{n=0}^{\infty} U_n\ (z)$$

is the solution of second kind corresponding to $ce_2(z,\ q)$. It is evidently the solution $i\eta_2(z,\ q)$.

Again, if we proceed to construct the integral of the second kind corresponding to the integral $ce_3(z, q)$, for which

$$A=9+4q^2-8q^3+\frac{13}{5}q^4+\dots\text{etc.},$$

we obtain, in this case,

$$U_0(z)=\sin 3z.$$

$$U_1(z)=\tfrac{1}{2}q\sin 5z-q\sin z,$$

$$U_2(z)=\tfrac{1}{10}q^2\sin 7z+\tfrac{1}{4}pq\sin 5z+q(p-q)\sin z,$$

$$U_3(z)=\tfrac{1}{90}q^3\sin 9z+\tfrac{7}{160}q^2\sin 7z+\tfrac{1}{8}q(p^2+\tfrac{2}{5}q^2)\sin 5z$$
$$+\tfrac{1}{3}q^2(5p-8q)z\cos 3z-q(p-q^2)\sin z,$$

. . . .

. . , ...

and

$$\Sigma\lambda_n=\tfrac{8}{3}pq^2(5p-8q)+(\tfrac{32}{5}q^5-\tfrac{10}{3}pq^4+\tfrac{64}{3}p^2q^3-\tfrac{28}{3}p^3q^2)+\dots,\text{ etc.}$$
$$=\tfrac{5}{24}\{4q^2-8q^3+\dots\}^2q^2-\tfrac{8}{3}\{4q^2-8q^3+\dots\}q^3+\tfrac{32}{5}q^5$$
$$-\tfrac{5}{3}\{4q^2-8q^3+\dots\}q^4+\tfrac{1}{3}\{4q^2-8q^3+\dots\}^2q^3$$
$$-\tfrac{7}{384}\{4q^2-8q^3+\dots\}^3q^2+\dots,\text{ etc.}$$

$= 0$, neglecting higher powers of q than the fifth.

Hence $\Sigma U_n(z)$ as obtained from above is the second solution corresponding to $ce_3(z, q)$ and is denoted by $in_3(z, q)$.

That $\Sigma\lambda_n$ vanishes, has been further verified in a few other cases

On the Convergence of the Series of Integrals of the Second Kind.[1]

12. The process of *term by term differentiation*, which we have carried out in art. 8, Chapter II, is permissible if the infinite series $\Sigma U_n(z)$ is an uniformly convergent series of analytic functions. It is, therefore, necessary for us to examine the solution $\Sigma U_n(z)$ more

[1] Dhar: *Bull. of the Cal. Math. Soc.*, Vol. X, 1921. Also *American Jour. of Math.*, Vol. XLV, No. 3, pp. 217-221.

closely with a view to study its convergency and when we have done it, the question of the convergency which has not yet been discussed, will be solved along with it.

The forms of $U_n(z)$, which are solutions of the differential equation

$$\frac{d^2y}{dz^2} + m^2 y = W_n(z),$$

will be of the following types :—

(*i*) when $n < m$

$$U_n(z) = {**}\sum_{r=1}^{n}{}' \beta_{n,r} \sin(m-2r)z + \sum_{r=1}^{n} \alpha_{n,r} \sin(m+2r)z$$

(*ii*) when $n = m$.

$$U_m(z) = {**}\sum_{r=1}^{m}{}' \beta_{m,r} \sin(m-2r)z + \sum_{r=1}^{m} \alpha_{m,r} \sin(m+2r)z + \delta_{n,0} \cos mz$$

(*iii*) when $n > m$, *i.e.*, when $n = m+\eta$ and $\eta \geq 1$

$$U_{m+\eta}(z) = {**}\sum_{r=1}^{m+\eta}{}' \beta_{m+\eta,r} \sin(m-2r)z + \sum_{r=1}^{m+\eta} \alpha_{m+\eta,r} \sin(m+2r)z$$

$$+ z\left\{\sum_{r=1}^{\eta}{}' \gamma_{\eta,r} \cos(m-2r)z + \sum_{r=0}^{\eta} \delta_{\eta,r} \cos(m+2r)z\right\}$$

Then since

$$\left(\frac{d^2}{dz^2} + m^2\right). U_{m+\eta+1}(z) = -8(p+2q\cos 2z)\, U_{m+\eta}(z)$$

$$- \lambda_{m+\eta+1}.\, z \cos mz - a_n q^n \sin mz.$$

** $\sum'$ means that the summation ceases at the greatest value of r, for which $r \leq \frac{m}{2}$.

we get by equating the co-efficients of $z \cos(m+2r)z$, $z \cos(m-2r)z$, $\sin(m+2r)z$, and $\sin(m-2r)z$, the following recurrence formulæ :—

$$(a) \quad r(m+r)\ \delta_{\eta+1,\,r}=2\ \{ p\,\delta_{\eta,r}+q\,(\delta_{\eta,r-1}+\delta_{\eta,r+1})\},$$

$$(r=1,2,3,\ldots\ldots\ldots)$$

$$(b) \quad r(r-m)\,\gamma_{\eta+1,\,r}=2\{p\gamma_{\eta,r}+q\,(\gamma_{\eta,\,r-1}+\gamma_{\eta,\,r+1})\},\ \left(r\leq\frac{m}{2}\right)$$

$$(c) \quad r(m+r)\,a_{m+\eta+1,r}+\tfrac{1}{2}\,(m+2r)\ \delta_{\eta+1,\,r}$$

$$=2\,\{pa_{m+\eta,\,r}+q(a_{m+\eta,r+1}+a_{m+\eta,\,r-1})\}$$

$$\left(\begin{matrix} r=1,\ 2,\ 3,\ \ldots\ldots\ldots \\ \eta\geq 1 \end{matrix}\right)$$

but when $n<m$

$$r\,(m+r)\,a_{n,\,r}=2\left\{pa_{n,\,r}+q\,(a_{n,\,r+1}+a_{n,\,r-1}),\right\}$$

$$(d) \quad r\,(r-m)\,\beta_{m+\eta+1,\,r}+\tfrac{1}{2}\,(m-2r)\,\gamma_{\eta+1,\,r}$$

$$=2\{\,p\beta_{m+\eta,\,r}+q(\beta_{m+\eta,\,r+1}+\beta_{m+\eta,\,r-1})\,\}.$$

$$\left(r\leq\ \frac{m}{2}\right)$$

but when $n<m$

$$r\,(r-m)\,\beta_{n+1,\,r}=2\{p\beta_{n,\,r}+q(\beta_{n,\,r+1}+\beta_{\eta,\,r+1})\,\}.$$

$$\left(r\leq\frac{m}{2}\right)$$

These formulæ hold universally with following restrictions :—

$$a_{n,\,r}=\beta_{n,\,r}=\gamma_{n,\,r}=\delta_{n,\,r}=o,\ \text{if}\ r>n\,.$$

and also $a_{n,o}=\beta_{n,o}=\gamma_{n,o}=0$, (whatever n is)

13. If we put

$$\sum_{\eta=r}^{\infty} \delta_{\eta,\, r} \equiv D_r, \quad \sum_{\eta=r}^{\infty} \gamma_{\eta,\, r} \equiv C_r,$$

then D_r and C_r give the sum of the co-efficients of $z \cos(m-2r)z$ and $z \cos(m-2r)z$ respectively in $\sum U_n(z)$.

Hence from (a) and (b) in art. 12, we get

$$r(m+r)D_r = 2\{pD_r + q(D_{r-1} + D_{r+1})\} \quad \ldots \quad (20)$$

$$r(r-m)C_r = 2\{pC_r + q(C_{r-1} + C_{r+1})\} \ldots \quad \ldots \quad (21)$$

From the forms of (20) and (21) it is evident that D_r and C_r are both convergent[1] and that $\lim\limits_{r=\infty} D_r = 0$, and $\lim\limits_{r=\infty} C_r = 0$

Similarly, putting

$$\sum_{n=r}^{\infty} a_{n,\, r} \equiv A_r \; ; \quad \sum_{n=r}^{\infty} \beta_{n,\, r} \equiv B_r,$$

which represent respectively the co-efficients of $\sin(m+2r)z$ and $\sin(m-2r)z$, occurring in $\sum U_n(z)$ we get from (c) and (d)

$$r(m+r)A_r + \tfrac{1}{2}(m+2r)D_r$$
$$= 2\{pA_r + q(A_{r-1} + A_{r+1})\} \quad \ldots \quad (22)$$

$$r(r-m)B_r + \tfrac{1}{2}(m-2r)C_r$$
$$= 2\{pB_r + q(B_{r-1} + B_{r+1})\} \quad \ldots \quad (23)$$

Writing

$$\omega_r = -2q/\{r(m+r) - 2p\}, \quad \omega'_r = -2q/\{r(r-m) - 2p\},$$

$$v_r = (m+2r)/2\{r(m+r) - 2p\},$$

$$v'_r = (m-2r)/2\{r(m+r) - 2p\},$$

we get from (22) and (23) respectively

$$v_r D_r + \omega_r A_{r-1} + A_r + \omega_r A_{r+1} = 0 \quad \ldots \quad \ldots \quad (24)$$

$$v'_r C_r + \omega'_r B_{r-1} + B_r + \omega'_r B_{r+1} = 0 \ldots \quad \ldots \quad (25)$$

[1] Whittaker and Watson: *Loc. cit.* pp. 415-416

Eliminating $A_1, A_2, A_3, \ldots\ldots A_{r-1}\ A_{r+1}\ldots\ldots$ from (24) we get

$$A_r = \frac{(-1)^r}{\Delta_n} \square_r \quad \ldots \quad \ldots \quad \ldots \tag{26}$$

where
$$\Delta_n \equiv \left| \begin{array}{cccccccc} 1 & \omega_1 & & & & & & \\ \omega_2 & 1 & \omega_2 & & & & & \\ 0 & \omega_3 & 1 & \omega_3 & & & & \\ 0 & 0 & \omega_4 & 1 & \omega_4 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right|$$

$$\square_r = \left| \begin{array}{ccccccccccc} v_1 D_1 + \omega_1, & 1, & \omega_1, & & & & & & & & \\ v_2 D_2, & \omega_2, & 1, & \omega_2, & & & & & & & \\ v_3 D_3, & 0, & \omega_3, & 1, & \omega_3, & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ v_{r-1} D_{r-1}, & 0 & 0 & 0 & 0 & \omega_{r-1}, & 1, & 0 & & & \\ v_r D_r, & 0 & 0 & 0 & 0 & 0 & \omega_r, & \omega_r, & 0 & & \\ v_{r+1} D_{r+1}, & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega_{r+1}, & 0 & \\ v_{r+2} D_{r+2}, & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{r+2}, & 1, & \omega_{r+2}, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right|$$

since A_n is taken to equal to unity.

Expanding the determinant $\square_r$ in terms of the elements of the first column, we get

$$A_r = \frac{(-1)^r}{\Delta_n} \left\{ (v_1 D_1 + \omega_1) M_1 + \sum_{k=2}^{\infty} v_k D_k M_k \right\} \ldots \tag{27}$$

where M's denote the first minors of the elements of the first column.

14. It will be easily seen that with $s \geq 1$

$$(i)\quad M_{r+s}=(-1)^{r+s-1}\,\omega_r\omega_{r+1}\ldots\ldots\omega_{r+s-1}\cdot\ \Delta_{r+s}\ \times$$

$$\left|\begin{array}{cccccc}
1 & \omega_1 & & & & \\
\omega_2 & 1 & \omega_2 & & & \\
0 & \omega_3 & 1 & \omega_3 & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \omega_{r-1} & 1
\end{array}\right|$$

where Δ_{r+s} stands for the infinite determinant

$$\left|\begin{array}{cccccccc}
1 & \omega_{r+s+1} & & & & & & \\
\omega_{r\times s+2} & 1 & \omega_{r+s+2} & & & & & \\
0 & \omega_{r+s+3} & 1 & \omega_{r+s+3} & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|,$$

and also when $k<r$

$$(ii)\quad M_k=(-1)^{k-1}\omega_{k+1}\omega_{k+2}\ldots\ldots\omega_r.\Delta_r.\times$$

$$\left|\begin{array}{cccccccc}
1 & \omega_1 & & & & & & \\
\omega_2 & 1 & \omega_2 & & & & & \\
0 & \omega_3 & 1 & \omega_3 & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \omega_{k-1} & 1 &
\end{array}\right|$$

and further

(*iii*) $M_r = (-1)^{r-1} \times$

$$\left|\begin{array}{ccccccccccc}
1 & \omega_1 & & & & & & & & & \\
\omega_2 & 1 & \omega_2 & & & & & & & & \\
0 & \omega_3 & 1 & \omega_3 & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & \cdot & \omega_{r-1} & 1 & 0 & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \omega_{r+1} & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega_{r+2} & 1 & \omega_{r+2} & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|$$

The infinite determinant Δ_r is convergent[1] whatever "r" is and further $\lim\limits_{r \to \infty} \Delta_r = 1$. Hence the M's are all finite and the series (27) is convergent and converges to a finite value, if "r" is finite; but it vanishes if "r" is made indefinitely large.

The same may be proved for B_r by means of the relation (25).

The series $\sum U_n(z)$ is clearly a series of analytical function of "z" and can now easily be proved to be uniformly convergent in any bounded domain of "z," so that term by term differentiation is permissible.[2]

Thus the series of integrals of the second kind, which we have obtained, are convergent.

The Construction of the Solutions by Integral Equations of the Second kind.[3]

14. The integral equations of the second kind which we have obtained in art. 7, Chap. I. can also be used to construct the solutions of the second kind of Mathieu's equation. This method, though

[1] Helge Von Koch: *Acta Mathematica*, Vol. XVI.
[2] Whittaker and Watson: *Modern Analysis*, pp. 91-92.
[3] Dhar: *Bull. Cal. Math. Soc.*, Vol. XVIII (1927).

sometimes laborious, is yet useful. We will first proceed to obtain the solutions $i\eta_m(z, q)$. The problem before us is, therefore, to obtain the solution $i\eta_m(z, q)$ by means of the integral equation

$$y(z)=B_1 \sin mz+\frac{1}{m}\int_0^z (a'+16q \cos 2t) \sin m(t-z).y(t)dt, \quad (28)$$

where

$$B_1=1+b_1 q+b_2q^2+\ldots, \; b_1, b_2 \ldots\ldots \text{ etc., being numerical}$$

$$\text{and } a'=\frac{32q^2}{m^2-1}-\frac{128(5m+7)q^4}{(m^2-1)^3(m^2-4)}-\ldots \text{ etc,}$$

$$i.e. =a^2q-a_4q^4-\ldots\text{etc,}$$

this expression for a' being the same as was required in obtaining the solution $ce_m(z,q)$ from the integral equation

$$y(z) = B \cos mz + \frac{1}{m}\int_0^z (a'+16q \cos 2t) \sin m(t-z)\, y(t)dt,$$

Art. 9, Chap. I; also art. 5, (11) Chap. II,

15. Now, suppose that

$$y(z)=f_0(z)+qf_1(z)+q^2f_2(z)+\ldots\ldots\text{etc.,}$$

where f_0, f_1 .... etc., are functions of z and do not contain q.

Substituting in the integral equation (28) and equating like power of q, we get

(i) $f_0(z)=\sin mz$,

(ii) $f_1(z)=b_1 \sin mz+\frac{1}{m}\int_0^z 16 \cos 2t.\sin m(t-z)f_0(t)dt,$

(iii) $f_2(z)=b_2\sin mz+\frac{a_2}{m}\int_0^z \sin m(t-z)f_0(t)\, dt$

$$+\frac{1}{m}\int_0^z 16\cos 2t \sin m\,(t-z)\,f_1\,(t)\,dt,$$

$$(iv)\quad f_3\,(z)=b_3\sin mz+\frac{a_3}{m}\int_0^z \sin m\,(t-z)\,f_1 dt$$

$$+\frac{1}{m}\int_0^z 16\cos 2t\sin m\,(t-z)\,f_2 dt,$$

....

....

We solve the above equations in succession and thus obtain $f_1, f_2,\ldots$ etc. The numerical constants $b_1, b_2,\ldots$ etc. are obtained from the condition that the forms of $f_1, f_2,\cdots$ must not contain $\sin mz$ as a term in them. Thus we obtain,

$$(a)\quad f_1\,(z)=\left(b_1-\frac{4}{m^2-1}\right)\sin mz+\frac{2\sin(m+2)z}{m+1}$$

$$-\frac{2\sin(m-2)z}{m-1}$$

$$\therefore\quad \text{if } b_1=\frac{4}{m^2-1}\text{, then } f_1\,(z)=\frac{2\sin(m+2)z}{m+1}$$

$$-\frac{2\sin(m-2)z}{m-1},$$

$$(b)\qquad f_2\,(z)=\frac{2}{(m+1)\,(m+2)}\sin(m+4)z$$

$$+\frac{2}{(m-1)(m-2)}\sin(m-4)\,z.$$

$$\text{where } b_2=\frac{4(m^4+4m^3-14m^2-16m+16)}{m^2(m^2-1^2)(m^2-2^2)}.$$

and so on.

[*Cf.* the method of arts. 6, 7, 8, of Chap. II.]

Proceeding as above we will obtain the solution $i\eta_m\ (z, q)$.

16. We proceed to show the method by working out a few particular cases.

Let us construct the integral $i\eta_1(z, q)$ by means of the integral equation

$$y(z)=B_1\sin z+\int_0^z (a'+16q\ cos\ 2t)\ \sin\ (t-z)y(t)dt,$$

where $\quad B_1=1+b_1q+b_2q^2+\ldots\ldots$etc.

and $\quad a'=-8q-8q^2-8q^3-\frac{8}{3}q^4+\ldots\ldots$etc.,

Taking $y(z)=f_0(z)+qf_1(z)+q^2f_2(z)+\ldots$

and substituting in the above integral equation, we get

$$f_0(z)+qf_1(z)+q^2f_2(z)+\ldots\ldots$$

$$=(1+b_1q+b_2q^2+\ldots)\sin z+\int_0^z \{-8q-8q^2+8q^3\ldots\ldots)$$

$$+16q\cos 2t\}.\sin (t-z)\ \{f_0+q\,f_1+\ldots\ldots\}\ dt.$$

Therefore equating like powers of q, we obtain.

(*i*) $f_0(z)=\sin z$.

(*ii*) $f_1(z)=b_2\sin z+\int_0^z(-8+16\cos 2t)\sin(t-z)f_0\ (t)\ dt,$

(*iii*) $f_2(z)=b_2\sin z-8\int_0^z\sin(t-z)\ f_0(t)dt$

$$+\int_0^z(-8+16\cos 2t)\ \sin\ (t-z)\ f_1(t)\ dt.$$

... ...

... ...

Solving the above in succession, we obtain.

(a) $f_0(z) = \sin z$,

(b) $f_1(z) = b_1 \sin z - 8z \cos z + \sin 3z + 5 \sin z$,

Choose $b_1 = -5$, so that $f_1(z)$ does not contain $\sin z$, then

$f_1(z) = \sin 3z - 8z \cos z$.

(c) $f_2(z) = (b_2 - \frac{26}{3}) \sin z + 5 \sin 3z + \frac{1}{3} \sin 5z - 8z \cos 3z$.

Now, if $b_2 = \frac{26}{3}$, then $f_2(z) = (\frac{1}{3} \sin 5z + 5 \sin 3z) - 8z \cos 3z$.

(d) $f_3(z) = (b_3 - 8\frac{1}{18}) \sin z + 24z \cos z + 8z \cos 3z - \frac{8}{3} z \cos 5z$

$$+ \frac{1}{18} \sin 7z + \frac{8}{3} \sin 5z - \frac{35}{3} \sin 3z.$$

$\therefore$ if $b_3 = 8\frac{1}{18}$. then $f_3(z) = z(24 \cos z + 8 \cos 3z - \frac{8}{3} z \cos 5z)$

$$+ (\frac{1}{18} \sin 7z + \frac{8}{3} \sin 5z - \frac{35}{3} \sin 3z), \text{ and so on.}$$

Hence the integral $i\eta_1(z, q)$ is given by the integral equation

$$y(z) = (1 - 5q + \frac{26}{3} q^2 + 8\frac{1}{18} q^3 + \ldots) \sin z$$

$$+ \int_0^z (a' + 16q \cos 2t) \sin (t - z)\, y(t)\, dt,$$

where $a' = -8q - 8q^2 + 8q^3 - \ldots\ldots$etc.

N. B.—The same integral $i\eta_1(z, q)$ can also be obtained by solving the integral equation

$$y(z) = (1 - 4q + 12q^3 + \ldots) \sin z + \int^z (a' + 16q \cos 2t) \sin (t - z)\, y(t)\, dt.$$

a' having the same value as above.

17. If we proceed to construct the integral $i\eta_2(z, q)$ from the integral equation

$$y(z) = B_1 \sin 2z + \frac{1}{2} \int_0^z (a' + 16q \cos 2t) \sin 2(t - z)\, y(t)\, dt,$$

where $a' = \frac{80}{3} q^2 - \frac{6104}{27} q^4 + \ldots\ldots$etc., we obtain according to the above procedure.

(*i*) $f_0\ (z) = \sin\ 2$

(*ii*) $f_1\ (z) = (b_1 - \frac{4}{3}) \sin\ 2z + \frac{2}{3} \sin\ 4z$

$\therefore$ if $b_1 = \frac{4}{3},\ f_1(z) = \frac{2}{3} \sin\ 4z$

(*iii*) $f_2\ (z) = (b_2 - \frac{9}{2}) \sin\ 2z + 8z \cos\ 2z + \frac{1}{5} \sin\ 6z$

$\therefore$ if $b_2 = \frac{9}{2},\ f_2\ (z) = 8z \cos\ 2z + \frac{1}{6} \sin\ 6z$

(*iv*) $f_3(z) = (b_3 + 9\frac{23}{135}) \sin\ 2z - 16z + \frac{16}{3} z \cos\ 4z + \frac{1}{45} \sin\ 8z - \frac{53}{27} \sin\ 4z.$

$\therefore$ if $b_3 = -9\frac{23}{135},\ f_3(z) = -16z + \frac{16}{3} z \cos\ 4z + \frac{1}{45} \sin\ 8z - \frac{53}{27} \sin\ 4z$

Hence $y\ (z) = \sin\ 2z + \frac{2}{3} q \sin\ 4z + q^2\ (8z \cos\ 2z + \frac{1}{6} \sin\ 6z)$

$$+ q^3 (-16z + \frac{16}{3} z \cos\ 4z + \frac{1}{45} \sin\ 8z - \frac{53}{27} \sin\ 4z) + \ldots\ldots$$

which is nothing but the integral $i\eta_2(z,\ q)$ and is given by the integral equation

$$y\ (z) = (1 + \tfrac{4}{3} q + \tfrac{9}{2} q^2 - 9\tfrac{23}{135}\ q^3 + \ldots) \sin 2z + \tfrac{1}{2} \int_0^z (a' + 16q \cos\ 2t)$$

$$\sin\ 2\ (t - z)\ y\ (t)\ dt.$$

CHAPTER III.

Properties of the Solutions.

1. Having now obtained the integrals of Mathieu's differential equation, we will proceed to investigate some of the properties of these functions. Of these some are very useful in physical problems :—*viz.*, the problem of vibration of an elliptic membrane, the problem of diffraction by an elliptic cylinder and in several others. Before proceeding further, let us summarise what we have done in the preceding pages.

It will be seen that if in the equations (7) and (8), Chap. I, we put

$$32q=-h^2p^2/c^2, \quad \text{and} \quad A=a+16q$$

they reduce to

$$\frac{d^2G}{d\eta^2}+(A+16q\cos 2\eta)G=0, \qquad \ldots \quad (1)$$

$$\frac{d^2F}{d\xi^2}-(A+16q\cosh 2\xi)F=0. \qquad \ldots \quad (2)$$

Solutions of (1) of the first kind are evidently

$$\left.\begin{array}{l} ce_0(\eta,q),\quad ce_1(\eta,q),\quad ce_2(\eta,q),\ldots\ldots ce_m(\eta,q),\ldots \\ \qquad\qquad se_1(\eta,q),\quad se_2(\eta,q),\ldots\ldots se_m(\eta,q),\ldots \end{array}\right\} \quad \ldots (1A)$$

and, therefore, those of (2) will be

$$\left.\begin{array}{l} ce_0(i\xi,q),\quad ce_1(i\xi,q),\quad ce_2(i\xi,q),\ldots\ldots ce_m(i\xi,q),\ldots \\ \qquad\qquad se_1(i\xi,q),\quad se_2(i\xi,q),\ldots\ldots se_m(i\xi,q),\ldots \end{array}\right\} \quad \ldots (2A)$$

The solutions of the second kind of (1), corresponding to those of the first kind, are

$$\left.\begin{array}{l} i\eta_0(\eta,q),\quad i\eta_1(\eta,q),\quad i\eta_2(\eta,q),\ldots\ldots i\eta_m(\eta,q)\ldots \\ \qquad\qquad j\eta_1,(\eta,q),\quad j\eta_2(\eta,q),\ldots\ldots j\eta_m(\eta,q),\ldots \end{array}\right\} \quad \ldots (1B)$$

and hence those of (2) will be given by

$$i\eta_0(i\xi, q),\quad i\eta_1(i\xi, q),\quad i\eta_2(i\xi, q),\ldots\ldots i\eta_m(i\xi, q),\ldots$$
$$j\eta_1(i\xi, q),\quad j\eta_2(i\xi, q),\ldots\ldots j\eta_m(i\xi, q),\ldots \qquad \ldots \text{(2B)}$$

Hence we can find the solutions of the equation

$$\frac{\partial^2 U}{\partial \xi^2}+\frac{\partial^2 U}{\partial \eta^2}+\frac{h^2p^2}{c^2}(\cosh^2\xi-\cos^2\eta)U=0. \qquad \ldots \quad (3)$$

The Mathieu Functions form an Orthogonal System.

2. Consider the two particular solutions

$$V_1=e^{\frac{ipz}{c}}\, ce_m(\eta, q)\, ce_m(i\xi, q)$$

$$V_2=e^{\frac{ipz}{c}}\, ce_n(\eta, q)\, ce_n(i\xi, q)$$

of Laplace's equation $\nabla^2 V=0$.

By Green's Theorem we have

$$\iint V_2\, \frac{\partial V_1}{\partial n}\, ds=\iint V_1\, \frac{\partial V_2}{\partial n}\, ds,$$

the surface integration being taken over the boundary of a closed space through which V_1 and V_2 are defined and

$$\frac{\partial V_1}{\partial n} \quad \text{and} \quad \frac{\partial V_2}{\partial n}$$

denote the space variations of V_1 and V_2 along the outward drawn normal to ds.

Let us take the closed space to be an elliptic cylinder bounded by the planes $z=0$, $z=-\infty$ and the curved surface $\xi=a$.

(i) On the elliptic plane $z=0$, and $\xi=a$ we have

$$V_2=ce_n(\eta, q)\, ce_n\, (i\xi, q)$$

and $$\frac{\partial V_1}{\partial n}=\left(\frac{\partial V_1}{\partial z}\right)_{z=0}=\frac{ip}{c}ce_m(\eta,q)ce_m(i\xi,q),$$

and $\iint V_2\,\frac{\partial V_1}{\partial n}$ gives

$$\frac{ip}{c}\int^{a}\int^{2\pi}\rho^2 ce_n(\eta,q)ce_n(i\xi,q)ce_m(\eta,q)ce_m(i\xi,q)d\xi d\eta$$

ρ^2 being$=h^2(\cosh^2\xi-\cos^2\eta)$, (see art. 1, Chap. I).

(*ii*) On the plane $z=-\infty$, $\xi=a$, we have $V_2=0$ and nothing is contributed by the integral.

(*iii*) For the curved surface $\xi=a$, we have

$$V_2=e^{\frac{ip}{c}z}ce_n(\eta,q)ce_n(i\xi,q)$$

$$\frac{V_1}{n}=\left(\frac{\partial V_1}{\partial s}\right)_{\xi=a}=\left(\frac{\partial V_1}{\partial\xi}\cdot\frac{d\xi}{ds}\right)_{\xi=a}$$

$$=\frac{i}{\rho_0}e^{\frac{ip}{c}z}ce_m(\eta,q)ce_m'(ia,q)$$

ρ_0 being $=h\sqrt{\cosh^2a-\cos^2\eta}$,

and the part contributed by the integral on the left is

$$\int_{-\infty}^{0}\int_{0}^{2\pi}\frac{1}{\rho_0}e^{\frac{2ip}{c}z}ce_n(\eta,q)ce_n(ia,q)ce_m(\eta,q)ce'_m(ia,q)\rho_0 dz d\eta$$

$$\frac{c}{2ip}\cdot e_n(ia,q)ce_m'(ia,q)\int^{2\pi}ce_n(\eta,q)ce_m(\eta,q)d\eta.$$

Working out the other side in a similar way, we obtain on simplification,

$$\{ce_n(ia,q)ce_m'(ia,q)-ce_n'(ia,q)ce_m(ia,q)\}\int_0^{2\pi}ce_n(\eta,q)ce_m(\eta,q)d\eta=0$$

$$\therefore\ \text{if } n\neq m,\quad\text{then }\int^{2\pi}ce_n(\eta,q)ce_m(\eta,q)d\eta=0,\quad\dots\quad(4)$$

but if $n=m$, then

$$\int_0^{2\pi} [ce_n(\eta, q)]^2 d\eta \neq 0 \quad \text{but} \quad = \text{a constant quantity.}$$

In the same way we can prove the orthogonality of

$$ce_n(\eta, q) \cdot se_m(\eta, q) \; ; \quad se_n(\eta, q) \cdot se_m(\eta, q), \cdots \text{etc.}$$

Integral Equations Connected with the Mathieu Functions.

3. Prof. Whittaker[1] has obtained certain homogeneous integral equations connected with the Elliptic Cylinder functions and has shown how these integral equations may be utilised to construct these cylindrical functions. Here we will proceed to obtain a general form[2] of the integral equation connected with these functions and will show that the Whittaker's types are but particular cases of that general form, obtained by taking special forms of the Kernel.

It is well-known that the solution of the wave-equation

$\left[cf. \text{ (5), Chapter I ; } k=\frac{p}{c} \right]$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+k^2 u=0, \qquad \dots \quad (5)$$

for *circular* bodies leads to circular cylindrical functions or Bessel's functions. Thus, for a system of circular transformation given by

$$x=r\cos\phi, \qquad y=r\sin\phi,$$

the differential equation (5) reduces to

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}+k^2 u=0, \qquad \dots \quad (6)$$

[1] E. T. Whittaker : *Int. Congress of Mathematicians*, 1912. Whittaker & Watson : *Modern Analysis*, pp. 407-412 (Third Edition).

[2] Dhar : " On Some Integral Equations connected with Elliptic Cylinder Functions," *Jour. of Dept. of Science* (Cal. Univ.), Vol. III, Feb., 1922, pp. 251-256.

whose solutions are given by

$$J_0(kr),\ J_1(kr)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\phi,\ J_2(kr)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}2\phi,\ \ldots\ J_n(kr)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}n\phi,\ldots, \quad (7)$$

and

$$Y_0(kr),\ Y_1(kr)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\phi,\ Y_2(kr)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}2\phi,\ \ldots\ Y_n(kr)\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}n\phi,\ \ldots \quad (8)$$

where Y_0, Y_1..........................etc. are the Neumann's functions.

In the same way, the solution of the same equation (1) for *elliptic* bodies leads to Elliptic Cylinder Functions or Mathieu Functions. Thus, if we reduce (5) to elliptic co-ordinates by the transformation,

$$x=h\cosh\xi\cos\eta,\quad y=h\sinh\xi\sin\eta,$$

we get

$$\frac{\partial^2 U}{\partial\xi^2}+\frac{\partial^2 U}{\partial\eta^2}+h^2k^2(\cosh^2\xi-\cos^2\eta)U=0, \quad \ldots \quad (9)$$

which, again by the substitution $U=G(\eta).\,F(\xi)$, where G and F are exclusively functions of η and ξ, respectively, resolves itself into finding the solutions of the two equations :—

$$\frac{\partial^2 G}{\partial\eta^2}+(A+16q\cos 2\eta)G=0,$$

$$\frac{\partial^2 F}{\partial\xi^2}-(A+16q\cosh 2\xi)F=0,$$

where $32q=-h^2k^2$ and $A=a+16q$, an arbitrary constant.

The solutions of the equation (2) can be obtained by putting $i\xi$ for η in the solutions of (1). Now, for a given set of values of A, solutions of (5) have been obtained. They are given by (1A) and (1B), and those of (2) by (2A) and (2B), Chapter III, and the method by which their actual forms can be obtained has been already discussed in Chapters I and II.

4. Suppose $U(\xi,\eta)$ is an integral of the differential equation (9) other than those obtained above in (1A) and (1B) of art. 1 of Chap. III. Then if $\phi(\theta)$ be analytic in $(-\pi, \pi)$, the integral defined by

$$G(\eta)=\int_{-\pi}^{\pi} U(i\theta,\eta)\phi(\theta)\,d\theta,$$

will satisfy the differential equation (1)

$$\text{if} \int_{-\pi}^{\pi} \left(\frac{\partial^2}{\partial \eta^2} + A + 16q \cos 2\eta\right) U(i\theta, \eta)\phi(\theta)d\theta = 0,$$

$$\textit{i.e.} \text{ if} \int_{-\pi}^{\pi} \left(\frac{\partial^2 U}{\partial \eta^2} + 16q \cos 2\eta . U\right) \phi(\theta)d\theta + A\int_{-\pi}^{\pi} U(i\theta, \eta)\phi(\theta)d\theta = 0.$$

But on substituting $\xi = i\theta$, and $32q = -h^2k^2$ in (9) we obtain

$$\frac{\partial^2 U}{\partial \eta^2} + 16q \cos 2\eta . U = \frac{\partial^2 U}{\partial \theta^2} + 16q \cos 2\theta . U$$

Hence the integral will satisfy the differential equation (1)

$$\text{if} \int_{-\pi}^{\pi} \phi(\theta) \left(\frac{\partial^2 U}{\partial \theta^2} + 16q \cos 2\theta . U\right) d\theta + A\int_{-\pi}^{\pi} U(i\theta, \eta)\phi(\theta)d\theta = 0,$$

which, on integrating by parts, reduces to

$$\left[\frac{\partial U}{\partial \theta} . \phi(\theta) - U\frac{\partial \phi}{\partial \theta}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} U(i\theta, \eta) \left\{\frac{\partial^2 \phi}{\partial \theta^2} + (A + 16q \cos 2\theta)\ \phi\right\} d\theta = 0.$$

Now, if U and ϕ be such that

$U(i\pi, \eta) = U(-i\pi, \eta)$	$\phi(\pi) = \phi(-\pi)$
$\left[\frac{\partial}{\partial \theta} U(i\theta, \eta)\right]_{\theta=\pi} = \left[\frac{\partial}{\partial \theta} . U(i\theta, \eta)\right]_{\theta=-\pi}$	$\left[\frac{\partial \phi}{\partial \theta}\right]_{\theta=\pi} = \left[\frac{\partial \phi}{\partial \theta}\right]_{\theta=-\pi}$ (9A)

and further if ϕ be such that

$$\frac{\partial^2 \phi}{\partial \theta^2} + (A + 16q \cos 2\theta)\phi = 0,$$

then both the integral and the integrated parts vanish, and the integral gives the periodic solution of (1). Thus, we see that ϕ must also be a periodic elliptic cylinder function of θ, formed with the same constants A and q as G (η) itself; but there does not exist more than one distinct periodic solution of (1) with the same constant

A and q.[1] Hence $\phi(\theta)$ must be (save for a constant multiplier) what $G(\eta)$ becomes when η is replaced by θ, *i.e.*, it must be equal to $\lambda\, G(\theta)$. In particular, when U and ϕ are periodic, the conditions (9A) are satisfied. Hence, the periodic solution of (1) must satisfy the integral equation

$$G(\eta)=\lambda\int_{-\pi}^{\pi}U(i\theta,\eta)\,G(\theta)\,d\theta, \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

the Kernel $U(i\theta,\eta)$ being symmetrical in θ and η and also *periodic.*

As is apparent, the solutions of the integral equation will naturally depend upon the Kernel. Thus, according as it is even or odd periodic function of η, the integral equation will give us even or odd solutions. Again, it should be observed that for a given Kernel, the integral equation can have non-zero solutions when and only when λ has certain *characteristic* values, *i.e.*, *Eigenwerte*, corresponding to which the periodic solutions of (1) will be *Eigenfunktionen.*

5. Since the periodic solutions (1) as given in (1A) are solutions of the homogeneous integral equation (10), it follows from the theory of homogeneous integral equation that these solutions must satisfy the orthogonal property, *i.e.*,

$$(i)\ \int_{-\pi}^{\pi}G_m(\eta)\,G_n(\eta)\,d\eta=0. \qquad (m\neq n) \quad \ldots \quad (11)$$

$$(ii)\ \int_{-\pi}^{\pi}[G_m(\eta)]^2 d\eta=\text{const.}, \quad \ldots \quad (12)$$

Or in other words

$$\left.\begin{aligned}&\int_{-\pi}^{\pi}ce_m(\eta,q).\,ce_n(\eta,q)\,d\eta=0, && (m\neq n)\\ &\int_{-\pi}^{\pi}se_m(\eta,q).\,se_n(\eta,q)\,d\eta=0, && (m\neq n)\\ &\int_{-\pi}^{\pi}ce_m(\eta,q).\,se_{n}(\eta,q)\quad=0,\end{aligned}\right\} \quad \ldots \quad (13)$$

c. f. (4) Chap. III.

[1] The truth of this statement is restricted to small values of $|q|$ and the case when $|q|$ is large, is left open. *Cf.* Whittaker & Watson: Modern Analysis § 19. 2.

The above integral equation (10) is suggested by the following procedure :—

If U (ξ,η) is an integral of the differential equation (1) other than those hitherto obtained in (1A) and (2A), it must be possible to develop it in terms of the known solution given by (1A) and (2A). Hence

$$U(\xi,\eta)=\sum_{m=0}^{\infty} a_m G_m(\eta)\, G_m(i\xi), \qquad \ldots \qquad \ldots \quad (14)$$

Now, by multiplying both members of (14) by $G_m(\eta)$ and integrating between $-\pi$ and π, we obtain, by means of the known orthogonal properties of such functions

$$\int_{-\pi}^{\pi} U(\xi,\eta)\, G_m(\eta)\, d\eta = a_m G_m(i\xi) \int_{-\pi}^{\pi} [G_m(\eta)]^2\, d\eta,$$

which by (12) may be written as

$$G_m(\theta) = \lambda \int_{-\pi}^{\pi} U(i\theta,\eta)\, G_m(\eta)\, d\eta, \qquad \ldots \qquad \ldots \quad (15)$$

as $G(-\theta)=\pm\, G(\theta)$ and λ is any const. quantity.

6. To get the particular forms of the above homogeneous integral equation, as for instance to obtain what may be called, integral equations of Whittaker's types, we shall have only to take particular forms of the Kernel. The particular type of $G(\theta)$, to be selected from (*i*), (*ii*), (*iii*) and (*iv*) of (10) Chapter I, satisfying any integral equation, can be easily determined by examining how the Kernel behaves, first when '$-\eta$' is put for 'η' and secondly when '$\eta+\pi$' is put for 'η'. Thus if we take

$$\cos kx,\ \sin kx,\ x\cos ky,\ y\cos kx$$
$$\cos ky,\ \sin ky,\ x\sin ky,\ y\sin kx$$

as the forms of $U(x, y)$ which satisfy the differential equation (5) Chap. III, we may obtain the particular forms of the Kernel $U(i\theta,\eta)$ from them, first by substituting for $x=h\cosh\xi\cos\eta$ and $y=h\sinh\xi\sin\eta$ and then $i\theta$ for ξ. The solutions which will be associated with these types of Kernels, can best be given in a table which is appended herewith.

If, further, we take

$$J_0(kr),\ J_1\ (kr)\ \left.\begin{matrix}\sin\\ \cos\end{matrix}\right)\ \phi,\ J_2\ (kr)\ \left.\begin{matrix}\sin\\ \cos\end{matrix}\right)\ 2\phi,\ldots\ldots$$

$$J_n\,(kr)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right)\ n\phi,\qquad \ldots\ \text{etc.}$$

or

$$Y_0(kr),\ Y_1\ (kr)\ \left.\begin{matrix}\sin\\ \cos\end{matrix}\right)\ \phi,\ Y_2\ (kr)\ \left.\begin{matrix}\sin\\ \cos\end{matrix}\right)\ 2\phi,\ldots\ldots$$

$$Y_n\ (kr)\ \left.\begin{matrix}\sin\\ \cos\end{matrix}\right)\ n\phi,\ \ldots\ldots\ldots\ldots\text{etc.}$$

as the forms of U (x, y), the Kernels and the corresponding type can also be easily found out. We will illustrate our method by considering for instance the solution $J_n\ (kr)\ \cos n\phi$.

7. We can easily transform the solution in the following way :—

$$J_n\ (kr)\cos n\phi = \frac{J_n(kr)}{r^n}\ .\ r^n \cos n\phi$$

$$= \frac{J_n(kr)}{r^n}\ .\ r^n\ \Big\{\cos^n\phi - \frac{n(n-1)}{2\,!}$$

$$\cos^{n-2}\phi\,\sin^2\phi\ +\ \ldots\ldots\ldots\Big\},\qquad (16)$$

Now, if n is *even*, we get

$$J_n(kr)\cos n\phi = \frac{J_n(kr)}{r^n}\ \Big\{x^n - \frac{n(n-1)}{2\,!}\ x^{n-2}y^2 +$$

$$\ldots\ldots\ldots\ldots \pm y^n\Big\}\ ,\qquad \ldots\ (17)$$

which when we express in terms of ξ, and η becomes

$$\frac{J_n(kh\sqrt{\tfrac{1}{2}(\cosh 2\xi+\cos 2\eta)})}{\{\tfrac{1}{2}(\cosh 2\xi+\cos 2\eta)\}^{\frac{n}{2}}}\Big\{\cosh^n\xi\cos^n\eta-\frac{n(n-1)}{2!}$$

$$\cosh^{n-2}\xi\cos^{n-2}\eta\sinh^2\xi\cdot\sin^2\eta+\ldots\ldots\pm\sinh^n\xi\sin^n\eta\Big\}$$

Hence

$$U(\iota\theta,\eta)=\frac{J_n(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})}{\{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)\}^{\frac{n}{2}}}\Big\{\cos^n\theta\cos^n\eta+$$

$$\frac{n(n-1)}{2!}\cos^{n-2}\theta\cos^{n-2}\eta$$

$$\times\sin^2\theta\sin^2\eta+\ldots\ldots\ldots\ldots+\sin^n\theta\sin^m\eta\Big\}$$

Here if we put '$-\eta$' for η, the sign of the expression does not alter. Similarly, when we put '$\eta+\pi$' for η, then also the sign does not change. Therefore, from the integral equation (10), G (η) must be such that it does not alter its sign when we write for η, '$-\eta$' or '$\eta+\pi$'. It will be easily seen from (10) Chapter I that $G(\eta)$ must be of the form $ce_{2r}(\eta, q)$.

But if n is *odd*, the expression (16) will give us

$$J_n(kr)\cos n\phi=\frac{J_n(kr)}{r^n}\Big\{x^n-\frac{n(n-1)}{2!}x^{n-2}y^2+\ldots\ldots\pm xy^{n-1}\Big\}$$

Hence

$$U(i\theta,\eta)=\frac{J_n(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})}{\{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)\}^{\frac{n}{2}}}\Big\{\cos^n\theta\cos^n\eta+\frac{n(n-1)}{2!}$$

$$\cos^{n-2}\theta\sin^2\theta\cos^{n-2}\eta\sin^2\eta+\ldots\ldots+\cos\theta\sin^{n-1}\theta\cos\eta\sin^{n-1}\eta\Big\}$$

Here[1] if we put for η, '$-\eta$' the sign of $U(\iota\theta, \eta)$ does not alter, but when we put '$\eta+\pi$' for 'η', the sign alters. Hence from (10), $G(\eta)$ must be such that it alters its sign when we put for 'η' '$\eta+\pi$' but does not alter its sign, when we put "$-\eta$" for η. Hence $G(\eta)$ must be an integral of the type $ce_{2r}(\eta, q)$ [see (10) Chapter I].

In the same way we can discuss other expressions and find the integrals associated with them. The results of this investigation can best be given in a table, appended below :—

$U(\iota\theta,\eta)$	$G(\eta)$
$\cos(kh\cos\theta\cos\eta)$ $\cos(kh\sin\theta\sin\eta)$ $\left.\begin{matrix}J_{2m}\\ y_{2m}\end{matrix}\right)(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})\cos\left\{2m\tan^{-1}(i\tan\theta\tan\eta)\right\}$ $m=0, 1, 2, 3, \ldots\ldots$ etc.	$ce_{2r}(\eta,q)$
$\sin(kh\cos\theta\cos\eta)$ $\cos\theta\cos\eta\cosh(kh\sin\theta\sin\eta)$ $\left.\begin{matrix}J_{2m+1}\\ y_{2m+1}\end{matrix}\right)(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})\cos\left\{(2m+1)\tan^{-1}(\iota\tan\theta\tan\eta)\right\}$ $m=0, 1, 2, 3, \ldots\ldots$ etc.	$ce_{2r+1}(\eta,q)$
$\sinh(kh\sin\theta\sin\eta)$ $\sin\theta\sin\eta\cos(kh\cos\theta\cos\eta)$ $\left.\begin{matrix}J_{2m+1}\\ y_{2m+1}\end{matrix}\right)(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})\sin\left\{(2m+1)\tan^{-1}(\iota\tan\theta\tan\eta)\right\}$ $m=0, 1, 2, 3, \ldots\ldots$ etc.	$se_{2r+1}(\eta,q)$
$\cos\theta\cos\eta\sinh(kh\sin\theta\sin\eta)$ $\sin\theta\sin\eta\sin(kh\cos\theta\cos\eta)$ $\left.\begin{matrix}J_{2m}\\ y_{2m}\end{matrix}\right)(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})\sin\left\{2m\tan^{-1}(\iota\tan\theta\tan\eta)\right\}$ $m=1, 2, 3, \ldots\ldots$ etc.	$se_{2r}(\eta,q)$

[1] Dhar: *Tôhoku Math. Jour.*, Vol. 24.

CHAPTER IV.

Expansion of the Mathieu Functions in a Series of Bessel's Functions.[1]

1. In solving physical problems, as for instance in diffraction problems, it is necessary to show that the solutions of Mathieu's equation vanish for large value of the argument. For this and for other consideration, we will gain great advantage by expressing these solutions in a series of Bessel's Functions. Many of the results given below, were given by Heine, Särchinger, Schubert, Sieger and others. Later on Dr. John Dougall has in a beautiful paper[2] discussed the solutions of Mathieu's equation in series of Bessel's Functions.

As was already said the various periodic solutions of Mathieu's equation can be expressed in a series of sines and cosines of multiples of η and, following Heine, can be classified into four groups (*cf.* Chap. I, 10) *viz.*,

$$\left.\begin{aligned}
&(i) \quad ce_{2r}(\eta,q) = a_{0,r} + \sum_{n=1}^{\infty} a_{n,r} \cos 2n\eta \\
&(ii) \quad ce_{2r+1}(\eta,q) = \sum_{n=0}^{\infty} \beta_{n,r} \cos (2n+1)\eta, \\
&(iii) \quad se_{2r+1}(\eta,q) = \sum_{n=0}^{\infty} \gamma_{n,r} \sin (2n+1)\eta, \\
&(iv) \quad se_{2r}(\eta,q) = \sum_{n=1}^{\infty} \delta_{n,r} \sin 2n\eta,
\end{aligned}\right\} \quad \ldots \quad (1)$$

with the following recurrence formulae for finding the co-efficients, *viz.*,

$$(i) \quad a_{n+1,r} + a_{n-1,r} = \frac{4n^2 - A}{8q} a_{n,r},$$

$$(ii) \quad \beta_{n+1,r} + \beta_{n-1,r} = \frac{(2n+1)^2 - A}{8q} \beta_{n,r},$$

$$(iii) \quad \gamma_{n+1,r} + \gamma_{n-1,r} = \frac{(2n+1)^2 - A}{8q} \gamma_{n,r},$$

$$(iv) \quad \delta_{n+1,r} + \delta_{n-1,r} = \frac{4n^2 - A}{8q} \delta_{n-1,r},$$

[1] Dhar : "*The Jour. of the Indian Math. Soc.,*" Vol. XVI, 1926.

[2] J. Dougall : *Proc. Edin. Math. Soc.*, Vol. XXXIV, pp. 191-196.

As has been already said, the above series converge only for definite values of the parameter A of the differential equation and have for other values, no meaning. Now for convergence, it is necessary that $\lim_{n \to \infty} \alpha_n = 0$, and three similar relations should hold. As the recurrence formulae will give expressions for α_n, β_n, etc., in a series of rational and integral functions of A, the above conditions for convergence will furnish equations for determining A. Instead of proceeding as above, the special values of A for which the solutions exist have been found out by other means.[1]

Assuming that the special values of A for which the solutions exist have been found out and substituted in the recurrence formulae, we will, following the method of Dougall,[2] find the form of $\alpha_{n,r}$ only.

2. The special value of the parameter A, for which the solution $ce_{2r}(\eta, q)$ has been constructed, is

$$(2r)^2 + \frac{32q^2}{4r^2-1} - \frac{2^7(20r^2+7)q^4}{(4r^2-1)^3(4r^2-4)} - \dots \text{ etc.}$$

which we may denote by $(2R)^2$.

Further, if we write $2q = \lambda^2$, the recurrence-formula (1) becomes

$$\alpha_{n+1,r} + \alpha_{n-1,r} = \frac{n^2 - R^2}{\lambda^2} \alpha_{n,r}$$

Hence by Dougall's formulae,[3] we obtain

$$\frac{\alpha_{n,r}}{\alpha_{0,r}} = \frac{\lambda^{2n}}{\Pi(n+R)\,\Pi(n-R)} \left\{1 - \lambda^4 A_n^{(1)} + \lambda^8 A_n^{(2)} + \lambda^{12} A_n^{(3)} + \dots\dots\right\}$$

[1] E. L. Mathieu : *Liouville's Jour.* XIII.
Whittaker : *Fifth International Congress of Math.*, *1912.*
E. Linsay Ince : *Proc. Edin Math. Soc.*, Vol. XXXIII, 1914-15.
S. C. Dhar : *American Jour. of Math.*, Vol. XLV, 1923.

[2] Dougall, *loc. cit.*

[3] J. Dougall, *loc. cit.*

where

$$A_n^{(q)} = \sum_{p_1=0}^{\infty} \sum_{p_2=2}^{\infty} \sum_{p_3=2}^{\infty} \cdots \sum_{p_q=2}^{\infty} a_{n+p_1}\, a_{n+p_1+p_2} \cdots a_{n+p_1+p_2 \ldots +p_q},$$

$$a_n = \frac{1}{\{(n+1)^2 - R^2\}\,\{(n+2)^2 - R^2\}}.$$

For approximate evaluation of the series for small values of q, these forms are suitable. We will, following Dougall, denote the right-hand side expression by $\phi_{2r}(n)$. Hence

$$ce_{2r}(\eta, q) = a_{0,r} \sum_{n=0}^{\infty} \phi_{2r}(n) \cos 2n\eta. \qquad \ldots \qquad \ldots \quad (2)$$

3. (*i*) If we take the kernel as $\cos (kh \cos \theta \cos \eta)$, then by (10) Chap. III we shall obtain the integral equation

$$ce_{2r}(\eta, q) = \lambda_r \int_{-\pi}^{\pi} \cos (kh \cos \theta \cos \eta)\, ce_{2r}(\theta, q)\, d\theta$$

$$= \lambda_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} \cos (kh \cos \theta \cos \eta) \cos 2n\theta \, d\theta.$$

The value of λ_r can be obtained at once by writing $\eta = \frac{\pi}{2}$ in the above equation, *i.e.*,

$$ce_{2r}\left(\frac{\pi}{2}, q\right) = \lambda_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} \cos 2n\theta \, d\theta$$

$$= \lambda_r \,.\, a_{0,r} \,.\, 2\pi.$$

$$\therefore \quad \lambda_r = \frac{1}{2\pi} \cdot \frac{ce_{2r}\left(\frac{\pi}{2}, q\right)}{a_{0,r}}$$

Hence

$$ce_{2r}(\eta, q) = ce_{2r}\left(\frac{\pi}{2}, q\right) \cdot \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{a_{n,r}}{a_{0,r}} \times$$

$$\int_{-\pi}^{\pi} \cos(kh\cos\theta\cos\eta)\cos 2n\theta\, d\theta.$$

But by a modification of Bessel's integral, we have

$$J_{2n}(z) = \frac{(-1)^n}{2\pi}\int_{-\pi}^{\pi} \cos(z\cos\theta)\cos 2n\theta\, d\theta.$$

Therefore

$$ce_{2r}(\eta, q) = ce_{2r}\left(\frac{\pi}{2}, q\right) \sum_{n=0}^{\infty} (-1)^n \frac{a_{n,r}}{a_{0,r}} J_{2n}(kh\cos\eta)$$

$$= ce_{2r}\left(\frac{\pi}{2}, q\right) \sum_{n=0}^{\infty} (-1)^n \phi_{2r}(n) J_{2n}(kh\cos\eta)\ldots \qquad (3)$$

or $$ce_{2r}(i\xi, q) = ce_{2r}\left(\frac{\pi}{2}, q\right) \sum_{n=0}^{\infty} (-1)^n \phi_{2r}(n) J_{2n}(kh\cos\xi) \qquad (4)$$

(*ii*) If we take $U(i\theta, \eta) = \cosh(kh\sin\theta\sin\eta)$, we get

$$ce_{2r}(\eta, q) = \lambda'_r \int_{-\pi}^{\pi} \cosh(kh\sin\theta\sin\eta)\, ce_{2r}(\theta, q)\, d\theta$$

$$= \lambda'_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} \cosh(kh\sin\theta\sin\eta)\cos 2n\theta\, d\theta.$$

To obtain λ'_r put $\eta=0$ in the above relation and, we get

$$ce_{2r}(0, q) = \lambda'_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} \cos 2n\theta\, d\theta = \lambda'_r\, a_{0,r} . 2\pi$$

$$\lambda'_r = \frac{1}{2\pi} \cdot \frac{ce_{2r}(0, q)}{a_{0,r}}.$$

Hence

$$ce_{2r}(\eta, q)=ce_{2r}(0, q)\sum_{n=0}^{\infty}\frac{a_{n,r}}{a_{0,r}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cosh(kh\sin\theta\sin\eta)\cos 2n\theta\, d\theta$$

$$=ce_{2r}(0, q)\sum_{n=0}^{\infty}\frac{a_{n,r}}{a_{0,r}}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(ikh\sin\theta\sin\eta)\cos 2n\theta\, d\theta.$$

But $$J_{2n}(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(z\sin\theta)\cos 2n\theta\, d\theta.$$

$$\therefore\quad ce_{2r}(\eta, q)=ce_{2r}(0, q).\sum_{n=0}^{\infty}\frac{a_{n,r}}{a_{0,r}}.J_{2n}(ikh\sin\eta)$$

$$=ce_{2r}(0, q)\sum_{n=0}^{\infty}\phi_{2r}(n).J_{2n}(ikh\sin\eta) \qquad \ldots \quad (5)$$

Hence also

$$ce_{2r}(i\xi, q)=ce_{2r}(0, q)\sum_{n=0}^{\infty}\phi_{2r}(n)J_{2n}(-kh\sin\xi)$$

$$=ce_{2r}(0, q).\sum_{n=0}^{\infty}\phi_{2r}(n)J_{2n}(kh\sin\xi). \qquad \ldots \quad (6)$$

(*iii*) Further if we take $U(i\theta, \eta)=J_0(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})$. then

$$ce_{2r}(\eta, q)=\mu_r\int_{-\pi}^{\pi}J_0(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})\, ce_{2r}(\theta, q)\, d\theta$$

$$=\mu_r\sum_{n=0}^{\infty}a_{n,r}\int_{-\pi}^{\pi}J_0(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)})\cos 2n\theta\, d\theta.$$

Now, putting $\eta=0$ in the above relation, we obtain

$$ce_{2r}(0, q)=\mu_r \sum_{n=0}^{\infty} a_{n, r} \int_{-\pi}^{\pi} J_0(kh \cos\theta)\cos 2n\theta \, d\theta.$$

To find an expression for μ_r, we proceed as follows :—

We know that

$$J_0(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(z \sin\phi)\,d\phi,$$

or $$J_0(kh\cos\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(kh\cos\theta\sin\phi)\,d\phi.$$

Therefore

$$\int_{-\pi}^{\pi} J_0(kh\cos\theta)\cos 2n\theta\,d\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos(kh\cos\theta\sin\phi)\cos 2n\theta\,d\theta\,d\phi$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\int_{-\pi}^{\pi}\cos(kh\cos\theta\sin\phi)\cos 2n\theta\,d\theta$$

$$=\int_{-\pi}^{\pi}(-1)^n . J_{2n}(kh\sin\phi)\,d\phi$$

$$\therefore \quad ce_{2r}(0, q)=\mu_r\int_{-\pi}^{\pi}\sum(-1)^n a_{n, r} J_{2n}(kh\sin\phi)\,d\phi$$

$$=\mu_r \frac{a_{0, r}}{ce_{2r}\left(\frac{\pi}{2}, q\right)}\int_{-\pi}^{\pi} ce_{2r}\left(\frac{\pi}{2}-\phi,\ q\right)d\phi, \text{ from (3)}$$

i.e.,
$$\mu_r = \frac{1}{a_{0,r}} \cdot \frac{ce_{2r}(0, q) \cdot ce_{2r}\left(\frac{\pi}{2}, q\right)}{\int_{-\pi}^{\pi} ce_{2r}\left(\frac{\pi}{2}-\phi, q\right) d\phi}$$

Now, putting $i\xi$ for η, we obtain

$$ce_{2r}(i\xi, q) = \mu_r \sum_{n=0}^{\infty} a_{n,r} \int_{-\pi}^{\pi} J_0(kh\sqrt{\tfrac{1}{2}(\cos 2\theta + \cosh 2\xi)}) \cos m\theta \, d\theta$$

But by Neumann's addition theorem,

$$J_0(kh\sqrt{\tfrac{1}{2}(\cos 2\theta + \cosh 2\xi)})$$

$$= J_0\left(\frac{kh}{2}\sqrt{e^{2\xi} + 2\cos 2\theta + e^{-2\xi}}\right)$$

$$= \sum_{m=0}^{\infty} (-1)^m \epsilon_m J_m(\tfrac{1}{2}kh\, e^{\xi}) J_m(\tfrac{1}{2}kh\, e^{-\xi}) \cos 2m\theta,$$

where ϵ_m is the Neumann factor which is equal to 2 when $m \neq 0$ and is equal to 1, when m is zero.

Hence

$$ce_{2r}(i\xi, q) = \mu_r \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,r} (-1)^m \epsilon_m J_m(\tfrac{1}{2}kh\, e^{-\xi}) J_m(\tfrac{1}{2}kh\, e^{\xi}) \int_{-\pi}^{\pi} \cos 2n\theta \cos 2m\theta \, d\theta.$$

But $\int_{-\pi}^{\pi} \cos 2m\theta \cos 2n\theta \, d\theta = 0$, if $m \neq n$ and equal to π, when $m = n$.

Therefore

$$ce_{2r}(i\xi, q)=\mu_r.\ 2\pi \sum_{n=0}^{\infty} (-1)^n_r\ a_{n,r}\ J_n\left(\frac{kh}{2}e^{\xi}\right)J_n\left(\frac{kh}{2}e^{-\xi}\right) \qquad (7)$$

$$=\frac{ce_{2r}(0, q)\ ce_{2r}\left(\frac{\pi}{2}, q\right)}{\frac{1}{2\pi}\int_{-\pi}^{\pi} ce_{2r}\left(\frac{\pi}{2}-\phi, q\right)d\phi}\times$$

$$\sum_{n=0}^{\infty} (-1)^n\ \phi_{2r}(n)J_n\left(\frac{kh}{2}e^{\xi}\right)J_n\left(\frac{kh}{2}e^{-\xi}\right) \qquad \ldots\ (8)$$

(*iv*) Again by taking

$$U(i\theta,\ \eta)=J_2[\ kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)}\ ]\cos 2\phi,$$

we obtain

$$ce_{2r}(\eta, q)=\mu'_r\int_{-\pi}^{\pi} J_2.\cos 2\phi.\ ce_{2r}(\theta, q)\ d\theta.$$

$$=\mu'_r\sum_{n=0}^{\infty}\int^{\pi} a_{n,r}\ J_2.\cos 2\phi.\cos 2n\theta\ d\theta$$

$$\therefore\quad ce_{2r}(i\xi, q)=\mu'_r\sum_{n=0}^{\infty} a_{n,r}\int_{-\pi}^{\pi} J_2(\tfrac{1}{2}kh\sqrt{e^{2\xi}+2\cos 2\theta+e^{-2\xi}})\times$$

$$\cos 2\phi\cos 2n\theta\ d\theta. \qquad \ldots\ (9)$$

Now. $J_2\ [\tfrac{1}{2}kh\sqrt{e^{2\xi}+2\cos 2\eta+e^{-2\xi}}\ ]\cos 2\phi$

$$=\frac{J_2[\tfrac{1}{2}kh\sqrt{e^{2\xi}+2\cos 2\eta+e^{-2\xi}}]}{\tfrac{1}{4}h^2(e^{2\xi}+2\cos 2\eta+e^{-2\xi})}\ r^2\cos 2\phi$$

$$=\frac{J_2(kr)}{r^2}\times h^2(\cosh 2\xi\cos^2\eta-\sinh^2\xi)$$

Now, by Gegenbauer's Addition Theorem,[1] we have

$$\frac{J_2(lr)}{(lr)^2}$$

$$=2^2\sum_{m=0}^{\infty}(-1)^m\,(m+2)\frac{J_{2+m}(\tfrac{1}{2}kh\ e^{\xi}\ )\ J_{2+m}(\tfrac{1}{2}kh\ e^{-\xi}\ )}{(\tfrac{1}{2}kh)^2}\ C_m^2\ (\cos 2\eta) \qquad \dots \quad (10)$$

where C_m^2 $(\cos 2\eta)$ denotes the co-efficient of a^m in the expansion of $(1-2a\ \cos\ 2\eta+a^2)^{-2}$ in ascending powers of a. This co-efficient is evidently seen to be equal to

$$(m+1).1.2\cos 2m\eta+m.2.2\cos\ 2(m-2)\eta$$

$$+(m-1).3.2\cos 2(m-4)\eta+ \qquad \dots \quad (11)$$

Hence from (9) and (10), we have

$$ce_{2,r}(i\xi,q)=\mu'_r\sum_{m=0}^{\infty}\ (-1)^m(2+m)J_{2+m}(\tfrac{1}{2}kh\ e^{\xi}\ )J_{2+m}(\tfrac{1}{2}kh\ e^{-\xi}\ )$$

$$\times\{\,a''_{m,r}\ \cosh 2\xi-a'_{m,r}\ \sinh^2\ \xi\,\} \qquad \dots \quad (12)$$

where $a''_{m,r}=\sum_{n=0}^{\infty} a_{n,r}\int_{-\pi}^{\pi}\cos^2\theta\cos 2n\theta\cdot C_m^2\ (\cos 2\theta)d\theta$

and $a'_{m,r}=\sum_{n=0}^{\infty} a_{n,r}\int_{-\pi}^{\pi}\cos 2n\theta\ C_m^2\ (\cos\ 2\theta)d\theta$

Now $a''_{m,r}=\sum_{n=0}^{\infty} a_{n,r}\int_{-\pi}^{\pi}\{\cos 2(n+1)\theta+2\cos 2n\theta$

$$+\cos 2(n-1)\theta\,\}\ C_m^2\ (\cos 2\theta)d\theta$$

[1] Watson: *Theory of Bessel's Functions*, p. 363.

$$=\tfrac{1}{2}\pi\{(m+1).\,1(a_{m+1,\,r}+2a_{m,\,r}+a_{m-1,\,r})$$
$$+m.\,2(a_{m-1,\,r}+2a_{m-2,\,r}+a_{m-3,\,r})$$
$$+(m-1).\,3(a_{m-3,\,r}+2a_{m-4,\,r}+a_{m-5,\,r})$$
$$+\quad \dots \quad \dots \quad \dots \}\qquad \dots \quad (13)$$

which is obtained by using the relation (11) and integrating.

Similarly

$$a'_{m,\,r}=\tfrac{1}{2}\pi\ \{(m+1)\ 1.\ a_{m,\,r}+m.2.a_{m-2,\,r}$$
$$+(m-1).\ 3.\ a_{m-4,\,r}+\dots\}\qquad \dots \quad (14)$$

Similar expansions of $ce_{2r}(i\xi,\,q)$ by using $J_3 \cos 3\phi$, $J_4 \cos 4\phi\dots$ etc. for the kernels can be obtained.

(v) Another but different form of the integral $ce_{2r}(i\xi,\,q)$ can be obtained, if we take the Kernel of the form

$$U(i\theta,\,\eta)=Y_0(kh\sqrt{\tfrac{1}{2}\,(\cos 2\theta+\cos 2\eta)}\,)$$

In that case for a particular value of λ, say λ'_{2r} we obtain

$$ce_{2r}(i\xi,\,q)=\lambda'_{2r}\int_{-\pi}^{\pi} Y_0(kh\sqrt{\tfrac{1}{2}\,(\cos 2\theta+\cosh 2\xi)})ce_{2r}(\theta,\,q)d\theta,$$

$$=\lambda'_{2r}.\int_{-\pi}^{\pi} Y_0\left(\frac{kh}{2}\sqrt{e^{2\xi}+2\cos 2\theta+e^{-2\xi}}\right)ce_{2r}(\theta,\,q)d\theta.$$

But by Addition Theorem,[1] we get

$$Y_0\left(\frac{kh}{2}\sqrt{e^{2\xi}+2\cos 2\theta+e^{-2\xi}}\right)$$

$$=Y_0\left(\frac{kh}{2}e^{\xi}\right)J_0\left(\frac{kh}{2}e^{-\xi}\right)+2\sum_{n=1}^{\infty}\left(-1\right)^{n}Y_n\left(\frac{kh}{2}e^{\xi}\right)$$
$$\times J_n\left(\frac{kh}{2}e^{-\xi}\right)\cos 2n\theta.$$

[1] Gray and Mathew : *Treatise on Bessel's Functions*, p. 92.

Now, substituting this expression for Y_0 and the series for $ce_{2r}(\theta, q)$ in the integral equation and simplifying, we get

$$ce_{2r}(i\xi, q)=\lambda'_{2r}.2\pi \sum_{n=0}^{\infty}(-1)^n\epsilon_n a_{n,r}\,Y_n\left(\frac{kh}{2}e^{\xi}\right)J_n\left(\frac{kh}{2}e^{-\xi}\right) \quad \ldots \quad (15)$$

It should be noticed in passing that the above expression (15) was obtained by Sieger as the expansion for the solution of the second kind of Mathieu's equation, which cannot be the case as is evident from above.

Expressions similar to (7) and (15) can also be obtained for the integrals $ce_{2r+1}(i\xi, q)$, $se_{2r+1}(i\xi, q)$ and $se_{2r}(i\xi, q)$ by taking the Kernel $U(i\theta, \eta)$ to be of the form

$$J_1\left(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)}\right) \cos\{\tan^{-1}(i\tan\theta\tan\eta)\},$$

$$J_1\left(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)}\right) \sin\{\tan^{-1}(i\tan\theta\tan\eta)\}$$

and $$J_2\left(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)}\right) \sin\{2\tan^{-1}(i\tan\theta\tan\eta)\}$$

or $$Y_1\left(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)}\right) \cos\{\tan^{-1}(i\tan\theta\tan\eta)\},$$

$$Y_1\left(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)}\right) \sin\{\tan^{-1}(i\tan\theta\tan\eta)\}$$

and $$Y_2\left(kh\sqrt{\tfrac{1}{2}(\cos 2\theta+\cos 2\eta)}\right) \sin\{2\tan^{-1}(i\tan\theta\tan\eta)\}$$

respectively.

Some of these expressions have been investigated by various writers. We give below some of them.[1]

$$ce_{2r+1}(i\xi, q)$$

$$=2\pi.\,\lambda_{2r+1}.\cosh\xi\sum_{n=0}^{\infty}\beta'_{n,r}\,J_{1+n}\left(\frac{kh}{2}e^{\xi}\right)J_{1+n}\left(\frac{kh}{2}e^{-\xi}\right) \quad \ldots \quad (16)$$

[1] Sieger : *loc. cit.*, Bd. 27.
Heine : *Kugelfunctionen*, p. 333.
Gray and Mathew : *loc. cit.*, p. 92.

where $\qquad \beta'_{n,\,r}=\{\beta_{0,\,r}-\beta_{1,\,r}+\beta_{2,\,r}-\ldots+(-1)^n\beta_{n,\,r}\}(n+1)$

$se_{2r+1}(i\xi,\,q)=$

$$2\pi.\,\mu_{2r+1}\sinh\xi\sum_{n=0}^{\infty}\delta'_{n,\,r}\,J_{1+n}\left(\frac{kh}{2}e^{\xi}\right)J_{1+n}\left(\frac{kh}{2}e^{-\xi}\right),\ \ldots \quad (17)$$

where $\qquad \delta'_{n,\,r}=\{\delta_{0,\,r}-\delta_{1,\,r}+\ldots(-1)^n\delta_{n,\,r}\}(n+1)$

$se_{2r}(i\xi,\,q)$

$$=2\pi.\,\mu_{2r}\sinh\xi\cosh\xi\sum_{n=0}^{\infty}\gamma'_{n,\,r}\,J_{2+n}\left(\frac{kh}{2}e^{\xi}\right)J_{2+n}\left(\frac{kh}{2}e^{-\xi}\right), \quad (18)$$

where $\qquad \gamma'_{n,\,r}=\{\gamma_{0,\,r}-\gamma_{1,\,r}+\ldots+(-1)^n\gamma_{n,\,r}\}(n+1).$

4. That the above expressions (4)...(18) satisfy the differential equation (2), art. 1, Chapter III, may be seen by adopting the procedure given by Dougall.[1] We will illustrate the method by taking only the expression (7) and showing that it satisfies the differential equation, thus :—

The expressions $J_n(\lambda e^{\xi})$ and $J_n(\lambda e^{-\xi})$, where $J_n(x)$ is a Bessel's function of the nth order, satisfy the equations

$$(i) \quad \frac{d^2y}{d\xi^2}+(\lambda^2e^{2\xi}-n^2)y=0$$

$$(ii) \quad \frac{d^2y}{d\xi^2}+(\lambda^2e^{-2\xi}-n^2)y=0$$

respectively.

Hence

$$\frac{d^2}{d\xi^2}\{J_n(\lambda e^{\xi}).J_n(\lambda e^{-\xi})\}+(\lambda^2e^{2\xi}+\lambda^2e^{-2\xi}-2n^2).$$
$$\{J_n(\lambda e^{\xi})J_n(\lambda e^{-\xi})\}$$
$$=-2\lambda^2J_n'(\lambda e^{\xi})J_n'(\lambda e^{-\xi}),$$

[1] Dougall : *Proc. Edin. Math. Soc.*, Vol. XXXIV, pp. 191-194.

or $$\left\{\frac{d^2}{d\xi^2}-(16q\cosh 2\xi-2n^2)\right\}.\left\{J_n(\lambda e^{\xi}).J_n(\lambda e^{-\xi})\right\}$$

$$=16qJ_n'(\lambda e^{\xi})J_n'(\lambda e^{-\xi})$$

where $$32q=-h^2k^2=-\frac{h^2p^2}{c^2} \text{ and } \lambda=\tfrac{1}{2}kh.$$

or $$\left\{\frac{d^2}{d\xi^2}-(16q\cosh 2\xi+A)\right\}.\left\{J_n(\lambda e^{\xi})\,J_n(\lambda e^{-\xi})\right\}$$

$$=16qJ_n'(\lambda e^{\xi})\,J_n'(\lambda e^{-\xi})+(2n^2-A)\,J_n(\lambda e^{\xi}).\,J_n(\lambda e^{-\xi}), \quad (19)$$

Now, by the recurrence formulae of Bessel's functions,

$$2J'_n=J_{n-1}-J_{n+1}$$

We have

$$4J'_n(\lambda e^{\xi}).J_n'(\lambda e^{-\xi})=\{J_{n-1}(\lambda e^{\xi})$$

$$-J_{n+1}(\lambda e^{\xi})\}\,\{J_{n-1}(\lambda e^{-\xi})-J_{n+1}(\lambda e^{-\xi})\}, \quad \ldots \quad (20)$$

Again, by the recurrence formulae,

$$\frac{2n}{ke^{\xi}}\cdot J_n(\lambda e^{\xi})=J_{n-1}(\lambda e^{\xi})+J_{n+1}(\lambda e^{\xi})$$

$$\frac{2n}{ke^{-\xi}}\cdot J_n(\lambda e^{-\xi})=J_{n-1}(\lambda e^{-\xi})+J_{n+1}(\lambda e^{-\xi})$$

we will have

$$\frac{4n^2}{\lambda^2}J_n(\lambda e^{\xi})\,J_n(\lambda e^{-\xi})$$

$$=\{J_{n-1}(\lambda e^{\xi})+J_{n+1}(\lambda e^{\xi})\}\,\{J_{n-1}(\lambda e^{-\xi})+J_{n+1}(\lambda e^{-\xi})\}, \quad \ldots \quad (21)$$

Hence from (20) and (21), we get

$$4J_n(\lambda e^{\xi})\,J'_n(\lambda e^{-\xi})+\frac{4n^2}{\lambda^2}\,J_n(\lambda e^{\xi})\,J_n(\lambda e^{-\xi})=2\,\{J_{n-1}(\lambda e^{\xi})$$

$$J_{n-1}(\lambda e^{-\xi})+J_{n+1}(\lambda e^{\xi})J_{n+1}(\lambda e^{-\xi})\}$$

or $8q\,J'_n(\lambda e^{\xi}).J'_n(\lambda e^{-\xi})-n^2J_n(\lambda e^{\xi})J_n(\lambda e^{-\xi})=4q\{J_{n-1}(\lambda e^{\xi})$

$$J_{n-1}(\lambda e^{-\xi})+J_{n+1}(\lambda e^{\xi})J_{n+1}(\lambda e^{-\xi})\} \quad \ldots \quad (22)$$

Therefore, from (19) and (22)

$$\left\{\frac{d^2}{d\xi^2}-(16q\cosh 2\xi+A)\right\}J_n(\lambda e^{\xi})J_n(\lambda e^{-\xi})$$

$$=(4n^2-A)J_n(\lambda e^{\xi})J_n(\lambda e^{-\xi})+8q\{J_{n-1}(\lambda e^{\xi})J_{n-1}(\lambda e^{-\xi})$$

$$+J_{n+1}(\lambda e^{\xi})J_{n+1}(\lambda e^{-\xi})\} \quad \ldots \quad (23)$$

Now, if we substitute

$$a_{0,r}\,J_0(\lambda e^{\xi})J_0(\lambda e^{-\xi})+\Sigma(-1)^n a_{n,r}\,J_n(\lambda e^{\xi})\,J_n(\lambda e^{-\xi}),$$

$$\lambda \text{ being } =\tfrac{1}{2}\,kh$$

in the differential equation

$$\frac{d^2y}{d\xi^2}-(16q\cosh 2\xi+A)y=0$$

we will have

$$a_{0,r}\left\{\frac{d^2}{d\xi^2}-(16q\cosh 2\xi+A)\right\}J_0(\lambda e^{\xi})J_0(\lambda e^{-\xi})\;+\sum_{1}^{\infty}(-1)^n a_{n,r}$$

$$\left[\frac{d^2}{d\xi^2}-(16q\cosh 2\xi+A)\right]J_n(\lambda e^{\xi})J_n(\lambda e^{-\xi})=0$$

which by (23) becomes

$$\Sigma(-1)^n a_{n,r}[(4n^2-A)J_n(\lambda e^{\xi})J_n(\lambda e^{-\xi})+8q\{J_{n-1}(\lambda e^{\xi})J_{n-1}(\lambda e^{-\xi})$$

$$+J_{n+1}(\lambda e^{\xi}).\,J_{n+1}(\lambda e^{-\xi})\}]=0.$$

Equating the co-efficients of $J_n(\lambda e^{\xi}).\ J_n(\lambda e^{-\xi})$, we get

$$(-1)^n(4n^2-A)a_{n,r}+(-1)^{n+1}\,8q\,a_{n+1,r}+(-1)^{n-1}\,8q\,a_{n-1,r}=0$$

or
$$a_{n+1,r}=\frac{1}{8q}(4n^2-A)a_{n,r}\ -a_{n-1,r}$$

These are exactly the recurrence formulae for the a's which we have obtained for $ce_{2r}(\eta, q)$ in (11) art. 3, Chapter I.

It will be further seen that the series is convergent, for by Cauchy's Criteria of Convergence, we have

$$\lim\left|\frac{u_{n+1}}{u_n}\right|=\lim\left|\frac{a_{n+1,r}}{a_{n,r}}\right|\left|\frac{J_{n+1}\left(\frac{kh}{2}\,e^{\xi}\right)}{J_n\left(\frac{kh}{2}\,e^{\xi}\right)}.\ \frac{J_{n+1}\left(\frac{kh}{2}\,e^{-\xi}\right)}{J_n\left(\frac{kh}{2}\,e^{-\xi}\right)}\right|$$

$$=\ \lim\left|\frac{a_{n+1,r}}{a_{n,r}}\right|\left|\frac{\frac{1}{2}\,kh\,e^{\xi}}{2(n+1)}.\ \frac{\frac{kh}{2}\,e^{-\xi}}{2(n+1)}\right|=0,$$

for when n is very large and x finite

$$J_n(x)=\frac{x^n}{2^n.n!}$$

5. Another fruitful method is to transform the equation (1) Chap. III by means of the relation.

$$\mu=\cos\eta.$$

The equation (1) is thereby transformed into

$$(1-\mu^2)\frac{d^2G}{d\mu^2}-\mu\frac{dG}{d\mu}+(A-16q+32q\mu^2)G=0$$

or
$$(1-\mu^2)\frac{d^2G}{d\mu^2}-\mu\frac{dG}{d\mu}+\{A+\tfrac{1}{2}h^2k^2-h^2k^2\mu^2\}G=0, \qquad \ldots \quad (24)$$

for $32q=-h^2k^2$.

Now, assume

$$G=\sum_{n=0}(-1)^n B_{n,r}\ J_{2n}(hk\mu)$$

Then substituting in the above equation (24), we get

$$\sum_{n=0}^{\infty}(-1)^n B_{n,r}\{(1-\mu^2)h^2k^2J''_{2n}-hk\mu J'_{2n}+[(A+\tfrac{1}{2}h^2k^2)-h^2k^2\mu^2]J_{2n}=0.$$

But from the properties of Bessel's functions :—

$$J''_n=\tfrac{1}{4}(J_{n-2}-2J_n+J_{n+2}),\ n\geq 2.$$

$$J''_0=\tfrac{1}{2}[J_2-J_0]$$

$$h^2k^2\mu^2J''_n+hk\mu J'_n+h^2k^2J_n=n^2J_n.$$

By the help of the above relations, the (above) expression is transformed into :—

$$\sum_{n=0}^{\infty}(-1)^n 4n^2 B_{n,r}\ J_{2n}-h^2k^2\sum_{n=0}^{\infty}\frac{1}{4}(-1)^n B_{n,r}\{J_{2n-2}-2J_{2n}+J_{2n+2}\}$$

$$-(\tfrac{1}{2}h^2k^2+A)\sum_{n=0}^{\infty}(-1)^n B_{n,r}\ J_{2n}=0$$

Equating the coefficients of J_{2n} to zero, we have

$$B_{n+1,r}+B_{n-1,r}=\frac{4}{h^2k^2}(A-4n^2)B_{n,r}.$$

Thus, it will be seen from a comparison of the recurrence formulae connecting $a_{n,r}$ in p. 52, that when we take A to be the value appropriate for the solution ce_{2r}, the coefficients $B_{0,r}$, $B_{1,r}$ ... $B_{n,r}$ are constants multiples of $a_{0,r}, a_{1,r}, \ldots a_{n,r}$.

Hence

$$ce_{2r}(\eta,q)=\lambda_r\sum^{\infty}(-1)^n a_{n,r}\ J_{2n}(kh\cos\eta).$$

a result, wnich we have already obtained by a different method (*cf.* 3, and 1, Chap. IV).

6. It vill be now easy to show that all the expressions (3), ... etc. (18) which we have obtained for the integrals of the first kind, vanish when ξ, is made infinitely great.

For, when x is very large, the asymptotic expansions for J_n and Y_n are given by the formulae

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\cos\left\{\frac{(2n+1)\pi}{4}-x\right\}$$

$$Y_n(x)=\sqrt{\frac{2}{\pi x}}(\log 2-\gamma)\cos\left\{\frac{(2n+1)\pi}{4}-x\right\}$$

$$-\sqrt{\frac{\pi}{2x}}\sin\left\{\frac{(2n+1)\pi}{4}-x\right\}.$$

Hence when ξ is made very large

$$J_n\left(\frac{kh}{2}e\right)=0,\quad \text{and}\quad Y_n\left(\frac{kh}{2}e^{\xi}\right)=0$$

Further, when x is very small, we have

$$J_n(x)=\frac{x^n}{2^n.n!}$$

Hence, when ξ is very large,

$$(i)\quad J_0\left(\frac{kh}{2}e^{-\xi}\right)=J_0(o)=1;\quad J_n\left(\frac{kh}{2}e^{\xi}\right)=J_n(o)=0,\ n>0.$$

$$(ii)\quad \cosh\xi\, J_1\left(\frac{kh}{2}e^{-\xi}\right)=\frac{e^{\xi}}{2}\cdot\frac{khe^{-\xi}}{\cdot 4}=\frac{kh}{8};$$

$$\cosh\xi\, J_{1+n}\left(\frac{kh}{2}e^{-\xi}\right)$$

$$=\frac{e^{+\xi}}{2}\left(\frac{kh}{2}\right)^{n+1}\cdot\frac{e^{-(n+1)\xi}}{2^{n+1}.(n+1)!}=0$$

$$(iii)\quad \sinh\xi\, J_1\left(\frac{kh}{2}e^{-\xi}\right)=\frac{kh}{8};\ \sinh\xi\, J_{n+1}\left(\frac{kh}{2}e^{-\xi}\right)=0,\ n\geq 1.$$

$$(iv)\quad \sinh\xi\cosh\xi\,J_2\left(\frac{kh}{2}e^{-\xi}\right)=\frac{k^2h^2}{32\times4};$$

and
$$\sinh\xi\cosh\xi\,J_{2+n}\left(\frac{kh}{2}e^{-\xi}\right)=0,\ n\geq1.$$

Also
$$(v)\quad \sinh^2\xi.\,J_2\left(\frac{1}{2}kh\,e^{-\xi}\right)=\tfrac{1}{4}\,\frac{k^2h^2}{4\cdot8}$$

and
$$\sinh^2\xi\,J_{2+n}\left(\frac{1}{2}kh\,e^{-\xi}\right)=0,\text{ when } n\geq1.$$

Therefore, if we examine the expressions (8), (12), (15) ... etc. ... (18) for very large values of ξ, we see that for $\xi=\infty$ all of them vanish, *i.e.*,

$$\left.\begin{array}{l}\lim\limits_{\xi=\infty} ce_{2r}(i\xi,q)=0,\ \lim\limits_{\xi=\infty} ce_{2r+1}(i\xi,q)=0,\\ \lim\limits_{\xi=\infty} se_{2r+1}(i\xi,q)=0,\ \lim\limits_{\xi=\infty} se_{2r}(i\xi,q)=0\end{array}\right\}\quad\ldots\quad(25)$$

N.B.—Before concluding this Chapter, it may be mentioned that the properties of Mathieu Functions have not yet been completely studied and there remain many things to be discovered in connection with their properties, before it can be brought in a line with functions of hypergeometric types. For instance, formulae for Mathieu functions corresponding to the recurrence formulae for functions of hypergeometric types, have yet to be discovered.

CHAPTER V.

Scattering of Electromagnetic waves by an Elliptic Cylinder.

1. The theory of diffraction as propounded by Huyghens is only approximately true, because it does not take into consideration the boundary conditions which the electric and magnetic forces of the waves must satisfy. A strictly regorous solution of the diffraction problem was first given by Sommerfeld[1] who found the diffraction of plane waves by the straight edge of an infinitely thin and perfectly conducting plane. After him various writers, *viz.*, Schwarzchild[2], J. J. Thomson, Rayleigh, and others have written on the subject. Later on, Prof. Sieger[3] and Aichi[4] both independently, have worked out the diffraction by an elliptic cylinder. We shall indicate below how elliptic cylinder functions were employed to solve the problem The problem which we propose is the following :—

A plane and polarised wave is allowed to fall upon the sides of a perfectly reflecting metallic cylinder of elliptic cross-section, we propose to find the diffraction produced by the obstacle.

2. The well-known differential equations for the propagation of electromagnetic waves, as given by Maxwell, reduces for elliptic cylinder co-ordinates, *viz.*,

$$x+iy=h\cosh(\xi+i\eta)$$

to the following equations :—

$$\left.\begin{aligned}
4\pi\sigma E_z+\epsilon\frac{\partial E_z}{\partial t}&=\frac{c}{\rho^2}\left[\frac{\partial\rho H_\eta}{\partial\xi}-\frac{\partial\rho H_\xi}{\partial\eta}\right]\\
4\pi\sigma E_\xi+\epsilon\frac{\partial E_\xi}{\partial t}&=\frac{c}{\rho}\frac{\partial H_z}{\partial\eta}-c\frac{\partial H_\eta}{\partial z}\\
4\pi\sigma E_\eta+\epsilon\frac{\partial E_\eta}{\partial t}&=c\frac{\partial H_\xi}{\partial z}-\frac{c}{\rho}\frac{\partial H_z}{\partial\xi}
\end{aligned}\right\}\quad\ldots\quad(1)$$

[1] A. Sommerfeld: *Math. Annalen*, Bd. 45 and 47.

[2] K. Schwarzchild: *Math. Ann.*, Bd. 55.

[3] B. Sieger: *Annalen der Physik*, Bd. 27.

[4] K. Aichi ; Tokyo Sugaku Buturigakkwai Kizi, 2nd Series, Vol. IV, No. 14.

and

$$\left.\begin{aligned}
\mu\frac{\partial H_z}{\partial t} &= \frac{c}{\rho^2}\left[\frac{\partial \rho E_\xi}{\partial \eta} - \frac{\partial \rho E_\eta}{\partial_\xi}\right] \\
\mu\frac{\partial H_\xi}{\partial t} &= c\frac{\partial E_\eta}{\partial z} - \frac{c}{\rho}\frac{\partial E_z}{\partial \eta} \\
\mu\frac{\partial H_\eta}{\partial t} &= \frac{c}{\rho}\frac{\partial E_z}{\partial \xi} - c\frac{\partial E_\xi}{\partial z}
\end{aligned}\right\} \quad \ldots \quad (2)$$

where E_ξ, E_η, E_z, H_ξ, H_η, H_z are the three components of the electric and magnetic forces in the three directions ξ, η, z respectively, c is the velocity of light, and also where

$$\rho^2 = h^2\ (\cosh^2\xi - \cos^2\eta),$$

(by (4) Chap. I.).

Firstly, assume that the electric force in the incident waves are parallel to the axis of the cylinder, which is taken as the z-axis, then

$$E_\xi = E_\eta = H_z = o,$$

and the set of equations (1) and (2) will reduce to

$$(i)\ 4\pi\sigma E_z + \epsilon\frac{\partial E_z}{\partial t} = \frac{c}{\rho^2}\left[\frac{\partial \rho H_\eta}{\partial \xi} - \frac{\partial \rho H_\xi}{\partial \eta}\right] \quad \ldots \quad (3)$$

$$(ii)\ \mu\frac{\partial H_\xi}{\partial t} = -\frac{c}{\rho}\frac{\partial E_z}{\partial \eta} \qquad (iii)\ \mu\frac{\partial H_\eta}{\partial t} = \frac{c}{\rho}\frac{\partial E_z}{\partial \xi} \quad \ldots \quad (4)$$

Now, eliminating H_ξ and H_η from the above equations, we get, when the medium is perfectly dialectric

$$\frac{\partial^2 E_z}{\partial \xi^2} + \frac{\partial^2 E_z}{\partial \eta^2} = \frac{\mu\rho^2\epsilon}{c^2}\frac{\partial^2 E_z}{\partial t^2} \quad \ldots \quad (5)$$

Secondly, assume that the magnetic force in the incident waves is parallel to the z-axis, or in other words that the electric force is in a plane perpendicular to the z-axis, then

$$H_\xi = H_\eta = E_z = 0.$$

and the equations (1) and (2) transform to

$$(i) \quad 4\pi\sigma E_\xi + \epsilon \frac{\partial E_\xi}{\partial t} = \frac{c}{\rho} \frac{\partial H_z}{\partial \eta},$$

$$(ii) \quad 4\pi\sigma E_\eta + \epsilon \frac{\partial E_\eta}{\partial t} = \frac{c}{\rho} \frac{\partial H_z}{\partial_\xi} \qquad \ldots \qquad (6)$$

$$(iii) \quad \mu \frac{\partial H_z}{\partial t} = \frac{c}{\rho^2} \left[\frac{\partial \rho E_\xi}{\partial \eta} - \frac{\partial \rho E \eta}{\partial \xi} \right] \qquad \ldots \qquad (7)$$

Hence when $\sigma = o$, we obtain on eliminating E_ξ and $E\eta$,

$$\frac{\partial^2 H_z}{\partial \xi^2} + \frac{\partial^2 H_z}{\partial \eta^2} = \frac{\mu\epsilon\rho^2}{c^2} \frac{\partial^2 H_z}{\partial t^2} \qquad \ldots \qquad (8)$$

Thus the two equations (5) and (8) are identical in form and it will do, if we solve only one.

3. Taking the first set of equations (4) and (5), we find that if a set of plane waves ψ_0 be incident on the cylinder the problem is to determine ψ which besides being solution of the differential equation (5), be such as will satisfy the boundary condition

$$\psi + \psi_0 = 0 \qquad \ldots \qquad (9)$$

on the boundary of the cylinder and

$$\psi = 0 \qquad \ldots \qquad (10)$$

at infinity. Then we can easily obtain H_ξ and $H\eta$ from E_z by the relations (4).

Now, let us suppose that the electric force E_z is a periodic function of time, and put

$$E_z = Ve^{i\omega_0 t}$$

Then the differential equation (5) transforms into

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}+h^2k^2(\cosh^2\xi-\cos^2\eta)\,V=0 \qquad \dots \qquad (11)$$

where $k^2=\frac{\mu\epsilon\omega_0{}^2}{c^2}$.

And if we put

$$V=F(\xi).\quad G(\eta)$$

then the above equation decomposes into the two ordinary equations

$$\frac{\partial^2 G}{\partial \eta^2}-(h^2k^2\cos^2\eta+A)G=0, \qquad \dots \qquad \dots \qquad (12)$$

$$\frac{\partial^2 F}{\partial \xi^2}+(h^2k^2\cosh^2\xi+A)F=0, \qquad \dots \qquad \dots \qquad (13)$$

which are the known equations for the elliptic cylinder functions (See Chapter I (7) and (8) whose solutions are obtained in previous Chapters and are given in (1A), (2A) and (1B), (2B) Chapter III, § 1.

4. For the incident wave, which we suppose to be propagated from the positive direction of the y axis to the negative direction of the elliptic cylinder given by $\xi=\beta$, let us put

$$\psi_0=Me^{i(ky+\omega_0 t)}, \qquad \dots \qquad (14)$$

where $\qquad \omega_0=Vk, \quad \& \quad V^2=c^2/\mu\epsilon.$ For air, $V=c$.

Here M is the amplitude of the wave, whose wave length is

$\lambda=\frac{2\pi}{k}$. If $kh<1$, then $\lambda>2\pi h$ and the incident wave will be such that its wave length will be greater than the distance between the foci of the confocal ellipses.

Now, $\qquad e^{iky}=\cos(kh\sinh\xi\sin\eta)+i\sin(kh\sinh\xi\sin\eta)$.

Therefore, since $\cos(kh \sinh \xi \sin \eta)$ and $\sin(kh \sinh \xi \sin \eta)$ satisfy the equation (11), they must be expansible in series of elliptic cylinder functions as :—

$$\cos(kh \sinh \xi \sin\eta) = \sum_{r=0}^{\infty} a_r \, ce_{2r}(\eta) ce_{2r}(i\xi),$$

$$\sin(kh \sinh \xi \sin\eta) = \sum_{r=1}^{\infty} b_r \, se_{2r+1}(\eta) se_{2r+1}(i\xi),$$

where $ce_{2r}(\eta)$ and $se_{2r+1}(\eta)$ are solutions of Mathieu's equation (12) and $ce_{2r}(i\xi)$ and $se_{2r+1}(i\xi)$ are the corresponding solutions of (13), such that

$$(i) \quad ce_{2r}(i\xi) = \frac{1}{2\pi} \int_0^{\xi} \cos(kh \sinh \xi) ce_{2r}(\eta) d\eta \qquad \dots \quad (14)$$

$$(ii) \quad \frac{1}{a_r} = \frac{1}{2\pi} \int_0^{2\pi} [ce_{2r}(\eta)]^2 d\eta \qquad \dots \quad (15)$$

And

$$(iii) \quad se_{2r+1}(i\xi) = \frac{1}{2\pi} \int_0^{\xi} \sin(kh \sinh \xi \sin\eta) se_{2r+1}(\eta) d\eta, \qquad (16)$$

$$(iv) \quad \frac{1}{b_r} = \frac{1}{2\pi} \int_0^{2\pi} [se_{2r+1}(\eta)]^2 d\eta \qquad \dots \quad (17)$$

Hence we obtain the development of the incident wave in terms of elliptic cylinder functions as

$$\psi_0 = Me^{i\omega t} \left\{ \sum_{r=0}^{\infty} a_r ce_{2r}(\eta) ce_{2r}(i\xi) \right.$$

$$\left. + i \sum_{r=0}^{\infty} b_r se_{2r+1}(\eta) se_{2r+1}(i\xi) \right\} \qquad \dots \quad (18)$$

The solutions $ce_{2r}(\eta)$ and $se_{2r+1}(\eta)$, which are known as Mathieu Functions have already been obtained in an earlier chapter.

The expressions for $ce_{2r}(i\xi)$, which were defined by (14) can be obtained by that integral or can be obtained by an expression of Heine, *viz.*,

$$ce_{2r}(i\xi)=a_{0,r}J_0(kh\sinh\xi)+\sum_{n=1}^{\infty} a_{n,r}J_{2n}(kh\sinh\xi), \quad \ldots \quad (19)$$

following the method of Art. 3 (*ii*), Chap. IV, thus we have

$$ce_{2r}(i\xi)=\frac{1}{2\pi}\int_0^{2\pi}\cos(kh\sinh\xi\sin\eta)\,ce_{2r}(\eta)d\eta$$

$$=\sum_{n=0}^{\infty} a_{n,r}\cdot\frac{1}{2\pi}\int_0^{2\pi}\cos(kh\sinh\xi\sin\eta)\cos 2n\eta\, d\eta$$

$$=\sum_{n=0}^{\infty} a_{n,r}\ J_{2n}(kh\sinh\xi)$$

cf. (6), Art. 3, Chap. IV.

In the same way, $se_{2r+1}(i\xi)$ can be obtained either by the integral (16), or by the expression,

$$se_{2r+1}(i\xi)=\sum_{n=0}^{\infty}\delta_{n,r}J_{2n+1}(kh\sinh\xi), \quad \ldots \quad (20)$$

which can be obtained by the method of Art. 3, Chap. IV, *viz.*,

$$se_{2r+1}(i\xi)=\frac{1}{2\pi}\int_0^{2\pi}\sin(kh\sinh\xi\sin\eta)se_{2r+1}(\eta)d\eta$$

$$=\frac{1}{2\pi}\sum_{n=0}^{\infty}\delta_{n,r}\int_0^{2\pi}\sin(kh\sinh\xi\sin\eta)\sin(2n+1)\eta\, d\eta$$

$$=\sum\delta_{n,r}J_{2n+1}(kk\sinh\xi).$$

Now, to find the diffraction produced by an elliptic cylinder given by $\xi=\beta$, corresponding to the incident wave given by (18), we will assume for the diffraction-wave, the expression

$$\psi = Me^{i\omega t}\left\{\sum_{r=0}^{\infty} A_r ce_{2r}(\eta)\, i\eta_{2r}(i\xi) + i\sum_{r=0}^{\infty} B_r\, se_{2r+1}(\eta)\, j\eta_{2r+1}(i\xi)\right\} \quad \ldots \quad (21)$$

where $i\eta_{2r}(i\xi)$ and $j\eta_{2r+1}(i\xi)$ are the solutions of the second kind corresponding to the solutions $ce_{2r}(i\xi)$ and $se_{2r+1}(i\xi)$ respectively.

Then by the condition that

$$[\psi_0 + \psi] = 0$$

when $\xi = \beta$, we obtain

$$A_r = -a_r \cdot \frac{ce_{2r}(i\beta)}{i\eta_{2r}(i\beta)} \quad \text{and} \quad B_r = -b_r \frac{se_{2r+1}(i\beta)}{j\eta_{2r+1}(i\beta)}, \quad \ldots \quad (22)$$

The coefficients A_r and B_r can now be calculated, as in the previous chapters methods were given for finding the integrals

$$ce_{2r}(i\xi), \quad se_{2r+1}(i\xi), \quad i\eta_{2r}(i\xi) \quad \text{and} \quad j\eta_{2n+1}(i\xi).$$

Case of a Small Elliptic Cylinder.

5. From physical considerations Lord Rayleigh [1] has investigated the scattering of sound waves by a small elliptic cylinder and hence he has, by suitable transformation, deduced the scattering of electromagnetic waves produced by such an obstacle. It will be presently shown that the results which we have obtained above, becomes simplified when we consider the case of an elliptic cylinder of small cross-section such that $h \cosh \xi$ and $h \sinh \xi$ are small. Further, we shall consider the scattering produced when the incident waves are of very large wave-lengths, so that $\omega (= kh)$ is very small.

Putting $\omega = kh$, we obtain the solutions of (12) by any one of the methods, indicated in Chapter I.

(*i*) $ce_0(\eta) = 1 - \dfrac{\omega^2}{8} \cos 2\eta + \dfrac{\omega^4}{512} \cos 4\eta + \ldots .$

(*ii*) $ce_2(\eta) = \cos 2\eta + \dfrac{\omega^2}{16}\left(1 - \dfrac{\cos 4\eta}{3}\right) + \dfrac{\omega^4}{6144} \cos 6\eta + .$

[1] Lord Rayleigh : *Scientific Papers IV*, p. 305.

(*iii*) $ce_4(\eta)=\cos 4\eta+\frac{\omega^2}{16}\left(\frac{1}{3}\cos 2\eta-\frac{1}{5}\cos 4\eta\right)-$

$$+\frac{\omega^4}{512}\left(\frac{1}{6}-\frac{1}{30}\cos 8\eta\right)+\ldots$$

(*iv*) $se_1\ (\eta)=\sin\ \eta-\frac{\omega^2}{32}\sin 3\eta+\frac{\omega^4}{1024}(\sin 3\eta+\frac{1}{3}\sin 5\eta)+\ldots$

(*v*) $se_3\ (\eta)=\sin\ 3\eta+\frac{\omega^2}{32}(\sin\ \eta-\frac{1}{2}\ \sin 5\eta)$

$$+\frac{\omega^4}{1024}\left(\frac{1}{10}\sin 7\eta-\sin\ \eta\right)+\ldots$$

(*vi*) $se_5\ (\eta)=\sin\ 5\eta+\frac{\omega^2}{32}\left(\frac{1}{2}\ \sin 3\eta-\frac{1}{3}\ \sin\ 7\eta\right)$

$$+\frac{\omega^4}{512\times 2}\left(\frac{\sin\ \eta}{4}+\frac{\sin\ 9\eta}{2}\right)+\ldots$$

and so on.

It is now easy to obtain the following values :—

$$\frac{1}{a_0}=\frac{1}{2\pi}\int_0^{2\pi}\{ce_0(\eta)\}^2 d\eta=1+\frac{\omega^4}{128},$$

$$\frac{1}{a_1}=\frac{1}{2\pi}\int_0^{2\pi}\{ce_2(\eta)\}^2 d\eta=\frac{1}{2}\left(1+\frac{19}{256\times 9}\ \omega^4\right),$$

$$\frac{1}{a_2}=\frac{1}{2\pi}\int_0^{2\pi}\{ce_4\ (\eta)\}^2 d\eta=\frac{1}{2}\left(1+\frac{34}{256\times 9\times 25}\ \omega^4\right),$$

$$\frac{1}{b_0}=\frac{1}{2\pi}\int_0^{2\pi}\{se_1\ (\eta)\}^2 d\eta=\frac{1}{2}\ \left(1+\frac{\omega^4}{1024}\right),$$

$$\frac{1}{b_1}=\frac{1}{2\pi}\int_0^{2\pi}\{se_3(\eta)\}^2 d\eta=\tfrac{1}{2}\left(1+\frac{5}{4096}\,\omega^4\right),$$

$$\frac{1}{b_2}=\frac{1}{2\pi}\int_0^{2\pi}\{se_5\ (\eta)\}^2 d\eta=\tfrac{1}{2}\left(1+\frac{13}{1024\times 36}\,\omega^4\right).$$

and so on.

Again, by the help of Heine's formulae (19) and (20) we can obtain the solutions of the equation (13), thus :—

$$(\alpha)\ ce_0\ (i\xi)=J_0\ (\omega \sinh\ \xi)-\frac{\omega^2}{8}\,J_2\ (\omega \sinh\ \xi)+\frac{\omega^4}{512}\times$$

$$J_4(\omega \sinh\ \xi)+\ldots=1-\frac{z^2}{4}+\frac{z^4}{64}-\frac{\omega^2 z^2}{64},\quad \text{when } z=\omega \sinh\ \xi.$$

$$(\beta)\quad ce_2\ (i\xi)=\frac{\omega^2}{16}-\frac{z^2}{8}-\frac{z^2\omega^2}{64}-\frac{z^4}{96},\ \text{etc.}$$

$$(\gamma)\quad ce_4\ (i\xi)=\frac{\omega^4}{512\times 6}+\frac{\omega^2 z^2}{48\times 8}+\frac{z^4}{16\times 24}$$

$$(\delta)\ se_1\ (i\xi)=J_1\ (z)-\frac{\omega^2}{32}\,J_3\ (z)+\ldots$$

$$=\frac{z}{2}\left(1-\frac{z^2}{8}+\frac{z^4}{192}-\frac{z^2\omega^2}{32\times 24}\right)$$

$$(\epsilon)\quad se_3\ (i\xi)=\frac{z}{16}\left(\frac{\omega^2}{4}+\frac{z^2}{3}-\frac{z^4}{48}-\frac{\omega^2 z^2}{32}-\frac{\omega^4}{128}\right)$$

$$(\phi)\ se_5\ (i\xi)=\frac{z}{256}\left(\frac{\omega^4}{32}+\frac{z^2\omega^2}{12}+\frac{z^4}{15}\right)$$

and so on.

6. Expression for $ce_n(i\xi)$ *i.e.*, integrals of (13) of the first kind, have been obtained above. The corresponding integrals $i\eta_n(i\xi)$ of the second kind, can be obtained by employing any one of the methods given in Chapter II. For purposes of their application to the present problem, however, and in order to obtain results in the form obtained by Lord Rayleigh, let us obtain the integrals $i\eta_n(i\xi)$ in the form given below. As we are considering a cylinder of small cross section, we need only find one or two terms.

It is well-known that if $y=v$ be a particular solution of the differential equation

$$\frac{d^2y}{dz^2}+Qy=0$$

then the most general solution of the above equation is given by

$$y=v\left(A+B\int \frac{1}{v^2}dz\right)$$

It follows that the solution of the second kind corresponding to the solution of the first kind $y=v$, will be given by

$$y=v\int_A^z \frac{1}{v^2}dz,$$

where A is an arbitrary constant, independent of z.

Hence when $v=ce_n\,(i\xi)$ we shall have

$$i\eta_n(i\xi)=ce_n(i\xi,)\int_A^\xi \frac{d\xi}{[ce_n(i\xi)]^2}$$

We shall, however, determine the lower limit from the condition that when sinh ξ or cosh ξ becomes $\frac{e^\xi}{2}$ while ω cosh ξ or ω sinh ξ is small, $i\eta_n\,(i\xi)$ shall coincide with Bessel's function of the second kind as defined by Hankel, *viz.*,

$$H_n\left(\frac{\omega e^\xi}{2}\right),$$

which was used by Lord Rayleigh in his Theory of Sound and which is defined as

$$H_n(z)=\frac{\pi}{2}Y_n(z)+\frac{i\pi}{2}J_n(z),$$

where $$\frac{\pi}{2}Y_n(z)=Y^{(n)}(z)-(\log 2-\gamma)\,J_n(z),$$

$Y^{(n)}(z)$ being Neumann's function and γ Euler's Const.

N.B.—It should be noticed that Nielson's definition of Hankel's function for imaginary argument which is denoted by Watson [1] by $H_n(z)$, is related to the above by

$$H_n(z)=\frac{i\pi}{2}H_n^{(2)}(z).$$

7. We will now proceed to obtain two expressions for the solutions of the second kind, according as ξ is small or large. For this the following formulae will be required:—

For z finite or small and n large

$$\left.\begin{aligned} J_n(z)&=\frac{z^n}{2^n n!} \\ H_n(z)&=-\frac{(n-1)!\,2^{n-1}}{z^n} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (23)$$

Also when $|z|$ is very large,

$$\left.\begin{aligned} J_n(z)&=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{2n+1}{4}\pi\right) \\ H_n(z)&=\sqrt{\frac{\pi}{2z}}.\ ie^{-i\left(z-\frac{2n+1}{4}\pi\right)} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (24)$$

[1] Watson: *Theory of Bessel's Function*, pp. 73-74.

Now, for small values of k we obtain the following expressions :—

$$i\eta_0'(i\xi)=ce_0(i\xi)\int_A^\xi \frac{d\xi}{[ce_0(i\xi)]^2}$$

$$=\left(1-\frac{\omega^2}{4}\sinh^2\xi\right)\int_A^\xi \frac{d\xi}{\left(1-\frac{\omega^2}{4}\sinh^2\xi\right)^2}$$

$$=\left(1-\frac{\omega^2}{4}\sinh^2\xi\right)\left(\xi-\frac{\omega^2}{4}\xi+\frac{\omega^2}{8}\sinh 2\xi+a+b\omega^2\right)$$

But

$$H_0\left(\frac{\omega}{2}e^\xi\right)=Y^{(0)}\left(\frac{\omega}{2}e^\xi\right)+\log\frac{i\epsilon}{2}\cdot J_0\left(\frac{\omega}{2}e^\xi\right), \text{ putting } \log\epsilon=\gamma,$$

$$=J_0\left(\frac{\omega}{2}e^\xi\right)\log\frac{\omega e^\xi}{2}+2\sum_1^\infty(-1)^{n-1}\frac{J_{2n}\left(\frac{\omega}{2}e^\xi\right)}{n}$$

$$+\log\frac{i\epsilon}{2}J_0\left(\frac{\omega}{2}e^\xi\right)$$

$$=J_0\left(\frac{\omega}{2}e^\xi\right)\left\{\log\frac{i\epsilon\omega}{4}+\xi+\frac{\omega^2e^{2\xi}}{16}\right\}$$

If the above two expressions become identical when

$$\omega\sinh\xi\longrightarrow\frac{\omega}{2}e^\xi,$$

$$\text{then } a=\log\frac{i\epsilon\omega}{4} \text{ and } b=0$$

$$\therefore\quad i\eta_0(i\xi)=\left(\xi+\log\frac{i\epsilon\omega}{4}\right)\left(1-\frac{\omega^2\sinh^2\xi}{4}\right)-\frac{\omega^2}{4}\left(\xi-\frac{\sinh 2\xi}{2}\right)$$

Similarly

$$i\eta_3(i\xi)=\frac{16}{\omega^2}\left(\cosh 2\xi-\frac{\omega^2}{4}\sinh^2\xi-\frac{\omega^2\sinh^4\xi}{6}\right)\left(\frac{1}{2}\tanh 2\xi+a\right)$$

But $$H_2\left(\frac{\omega e^{\xi}}{2}\right)=\frac{\omega^2 e^{2\xi}}{32}\left\{\log\frac{i\epsilon\omega}{4}+\xi-\frac{16\times 2^4}{\omega^4 e^{4\xi}}-\frac{32}{\omega^2 e^{2\xi}}\right\}$$

$\therefore \quad a=\frac{1}{2}$ and

$$i\eta_3(i\xi)\sim-\frac{8}{\omega^2}e^{-2\xi}$$

In the same way by the help of formula (23) we can find

$$i\eta_4(i\xi)=-\frac{67^2}{\omega^4}e^{-4\xi}\text{ , and so on.}$$

Again,

$$j\eta_1(i\xi)=\tfrac{1}{2}\ \omega\sinh\xi\left(1-\frac{\omega^2\sinh^2\xi}{8}\right)\int_A^{\xi}\frac{d\xi}{\frac{\omega^2}{4}\sinh^2\xi\left(1-\frac{\omega^2\sinh^2\xi}{8}\right)^2}$$

$$=\frac{2}{\omega}\sinh\xi\left(1-\frac{\omega^2\sinh^2\xi}{8}\right)\left[\tanh\xi+\frac{\omega^2}{4}\xi+a+b\omega^2\right]$$

Also

$$H_1\left(\frac{\omega e^{\xi}}{2}\right)=Y^{(1)}\left(\frac{1}{2}\omega e^{\xi}\right)+\log\frac{i\epsilon}{2}J_1\left(\frac{1}{2}\omega e^{\xi}\right)$$

$$=J_1\left(\frac{1}{2}\omega e^{\xi}\right)\left\{\log\frac{i\epsilon\omega}{4}+\xi-\frac{J_0\left(\frac{1}{2}\omega e^{\xi}\right)}{\frac{1}{2}\omega e^{\xi}\,J_1\left(\frac{1}{2}\omega e^{\xi}\right)}\right\}$$

$$=J_1\left(\frac{1}{2}\omega e^{\xi}\right)\left\{\log\frac{i\epsilon\omega}{4}+\xi-\frac{3}{4}-\frac{8}{\omega^2 e^{2\xi}}\right\}$$

$\therefore \quad a=-1$ and $b=\frac{1}{4}\log\frac{i\epsilon\omega}{4}-\frac{3}{16}$ and therefore

$$i\eta_1(i\xi)=-\frac{2}{\omega}\left[e^{-\xi}\left(1-\frac{\omega^2}{8}\sinh^2\xi\right)-\frac{\omega^2}{4}\sinh\xi\left(\xi+\log\frac{i\epsilon\omega}{4}-\frac{3}{4}\right)\right]$$

In the same way by the help of formula (23), we can obtain

$$j\eta_3(i\xi)\sim-\frac{64}{\omega^3}e^{-3\xi}$$

and so on.

For large values of ξ, we get by (24),

$$i\eta_0\ (i\xi)\sim\sqrt{\frac{\pi}{2kr}}\,ie^{-i\left(kr-\frac{\pi}{4}\right)}$$

$$i\eta_2\ (i\xi)\sim-\sqrt{\frac{\pi}{2kr}}\quad ie^{-i\left(kr-\frac{\pi}{4}\right)}$$

$$i\eta_4\ (i\xi)\sim\sqrt{\frac{\pi}{2kr}}\ ie^{-i\left(kr-\frac{\pi}{4}\right)}$$

... ...

$$j\eta_1(i\xi)\sim-\sqrt{\frac{\pi}{2kr}}\,.e^{-i\left(kr-\frac{\pi}{4}\right)}$$

$$j\eta_3(i\xi)\sim\sqrt{\frac{\pi}{2kr}}\quad e^{-i\left(kr-\frac{\pi}{4}\right)}$$

$$j\eta_5(i\xi)\sim-\sqrt{\frac{\pi}{2kr}}\quad e^{-i\left(kr-\frac{\pi}{4}\right)}$$

... ...

where $r=h\frac{e^{\xi}}{2}$. (*Cf.* art. 1 (3), Chap. 1.)

$$\text{Hence } A_0=-a_0\,\frac{ce_0(i\beta)}{i\eta_0(i\beta)}=-\frac{1}{\log\dfrac{i\epsilon\omega e^\beta}{4}}+\frac{\omega^2}{4}\frac{1}{\left(\log\dfrac{i\epsilon\omega e^\beta}{4}\right)^2}\times$$

$$\left(\beta-\frac{1}{2}\sinh 2\beta\right)$$

$$A_1=-a_1\,\frac{ce_2(i\beta)}{i\eta_2(i\beta)}=\frac{\omega^4}{64}\left(1-2\ \sinh^2\beta)\right)e^{2\beta}$$

... ... and so on

$$B_0=-b_0\,\frac{se_1(i\beta)}{j\eta_1(i\beta)}=\frac{\omega^2}{2}\sinh\beta e^\beta\ ,$$

and B_1, B_2,...will involve higher powers of ω than the second.

8. Let now, a and b represent the major and minor semi-axes respectively of the ellipse given by $\xi=\beta$, then by art. 1 (2), Chap. I,

$$a=h\cosh\beta,\text{ and } b=h\sinh\beta$$

$$\therefore\quad a+b=he^\beta\ .$$

Then at points on the plane whose distance r from the axis of the cylinder, is very large, we get,

$$\text{since } e^{+\frac{i\pi}{4}}=(i)^{+\frac{1}{2}},$$

$$ce_0(\eta)\ i\eta_0(i\xi)=-\sqrt{\frac{\pi}{2ikr}}\,e^{-ikr}\left\{1-\frac{k^2(a^2-b^2)}{8}\cos 2\eta\right\}$$

$$se_1(\eta)\ i\eta_1(i\xi)=-i\sqrt{\frac{\pi}{2ikr}}\,e^{-ikr}\left\{\sin\eta-\frac{k^2(a^2-b^2)}{3}\sin 3\eta\right\}$$

$$ce_2(\eta)\ i\eta_2(i\xi)=\sqrt{\frac{\pi}{2ikr}}\,e^{-ikr}\left\{\cos 2\eta+\frac{k^2(a^2-b^2)}{16}\left(1-\frac{\cos 4\eta}{3}\right)\right\}$$

... ...

Therefore the scattered waves are given by

$$\psi=\mathrm{M}e^{i\omega_0 t}\left[e^{-ikr}\times\sqrt{\frac{\pi}{2ikr}}\left\{\frac{1}{\log\frac{i\epsilon k(a+b)}{4}}+\frac{k^2 b(a+b)}{2}\sin\eta\right.\right.$$

$$\left.\left.-\frac{k^2(a^2-b^2)}{8\log\frac{i\epsilon k(a+b)}{4}}\cos 2\eta\right\}\right], \tag{25}$$

neglecting terms containing higher powers of k than the second.

If we confine our attention to the leading term, we have

$$\psi=\mathrm{M}e^{i\mathrm{V}kt}\;e^{-ikr}\sqrt{\frac{\pi}{2ikr}}\;\frac{1}{\log\frac{i\epsilon k(a+b)}{4}} \tag{26}$$

Following Lord Rayleigh, if we omit the imaginary part of the logarithm term, we obtain the realised scattered waves as

$$\psi=\mathrm{M}\sqrt{\frac{\pi}{2kr}}\;\frac{\cos k(\mathrm{V}t-r-\frac{1}{8}\lambda)}{\gamma+\log\{\frac{1}{4}k(a+b)\}} \tag{27}$$

corresponding to the incident wave

$$\psi_0=\mathrm{M}\cos k(\mathrm{V}t+\chi)$$

which coincides with the results obtained by Rayleigh.[1]

(a) For a circular cylinder, we take $a=b$, and then ψ from (26) reduces to

$$\psi=\mathrm{M}e^{ik(\mathrm{V}t-r)}\sqrt{\frac{\pi}{2ikr}}\cdot\frac{1}{\gamma+\log\frac{ika}{2}} \tag{28}$$

[1] Lord Rayleigh, Scientific Papers, Vol. IV, p. 317.

(*b*) For diffraction by an infinitely thin blade of length $2h$, we obtain the values of A_0, A_1...and B_0, B_1...when $\xi=0$, *i.e.*, when $\beta=0$. Hence we get

$$\psi = M.\sqrt{\frac{\pi}{2ikr}}\, e^{ik(Vt-r)}\left[\frac{1}{\log\frac{i\epsilon kh}{4}} + \frac{h^2k^2}{2}\sin\eta\right.$$

$$\left. - \frac{h^2k^2}{8\log\frac{i\epsilon kh}{4}}\cos 2\eta\right]$$

Confining to the leading term, we get

$$\psi = M\sqrt{\frac{\pi}{2ikr}}\, e^{ik(Vt-r)}\cdot\frac{1}{\gamma+\log\frac{ikh}{4}}. \tag{29}$$

a result which coincides with what is obtained by Rayleigh.[1]

[1] Lord Rayleigh, Scientific Papers, Vol. IV, p. 316.

N.B.—In this connection it may be mentioned that a similar problem of diffraction by a straight slit can be attacked by first finding out the diffraction produced by a small hyperbolic cylinder and then proceeding to the limiting case when that hyperbola reduces to a straight line. It is my intention to take up the discussion of this problem on some future occasion.

College of Science,
Nagpur University,
Nagpur.

ERRATA.

Page 9, 8th line, read $\frac{128\ 5m^2+7)q^4}{(m^2-1)^3(m^2-4)}$ for $\frac{128c(5m^2+7)q^4}{(m^2-1)^3(m^2-4)}$.

" 11, 10th line, " $u_1y_1+u_2y_2$ for $u_1y_1+uy_2$.

" 26, last but one line, " $\{8pq(6q+p)+\frac{8}{3}q^3\}\sin 3z$

for $8pq(6q+p)+\frac{8}{3}q^3\}\sin 3z$.

" 28, 8th line from below, " $\frac{128}{3}q^4$ for $\frac{128}{3}q^2$.

" 51, in the table,

2nd line, " $\cosh(kh\sin\theta\sin\eta)$ for $\cos(kh\sin\theta\sin\eta)$.

Pages 65, 67 and 69, in the title of the page,

read expansion for expantion

" 72, 7th line under (*ii*), read a minus sign after $=$.

" 80, 8th line, " $H_n^{(2)}(z)$ for $H_n(z)$.